1 Characterisation of Quasicoherent Sheaves by Distinguished Inclusions

[Section 13.3.3,[1]] (February 25, 2020)

Definition 1.1. *If X is a scheme, then an $\mathcal{O}_X$-module $\mathcal{F}$ is **quasicoherent** if for every affine open subscheme* $\operatorname{Spec} A \subset X$*,* $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ *for some A-module M.*

Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \subset X$ is a distinguished open subscheme of an affine open subscheme of X. Let $\phi : \Gamma(\operatorname{Spec} A, \mathcal{F}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ be the restriction map which factorises naturally as:

$$\begin{array}{ccccc} \Gamma(\operatorname{Spec} A, \mathcal{F}) & & \xrightarrow{\quad\phi\quad} & & \Gamma(\operatorname{Spec} A_f, \mathcal{F}) \\ & \searrow^{\otimes_A A_f} & & \nearrow_{\alpha} & \\ & & \Gamma(\operatorname{Spec} A, \mathcal{F})_f & & \end{array}$$

Proposition 1.1 (Very Important Ex, 13.3.D, [1])**.** *An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for every distinguished inclusion* $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$*,* $\Gamma(\operatorname{Spec} A, \mathcal{F})_f \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ *is an isomorphism.*

Proof. The only if direction is clear by definition. For the converse, let M be the A-module $M := \Gamma(\operatorname{Spec} A, \mathcal{F})$, the isomorphism shows that $\Gamma(\operatorname{Spec} A_f, \mathcal{F}) = M_f$. Since the distinguished open sets form a base, we have $\mathcal{F}|_{\operatorname{Spec} A} \cong \tilde{M}$, by [Theorem 2.5.1, [1]] which says that a sheaf defined on the base of X extends uniquely to a sheaf of X, up to unique isomorphism. □

Thus a quasicoherent sheaf is equivalent to the data of one module for each distinguished open subset such that the module over a distinguished open set Spec A_f is given by localizing the module. The next proposition shows this will be an easy criterion to check.

Proposition 1.2 (Important Ex 13.3.E, [1])**.** *Suppose X is a quasicompact and quasiseparated scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. The restriction map*

$$\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$$

(here X_f is the open subset of X where f doesn't vanish) is precisely localization In other words, there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ in the following commutative diagram

$$\begin{array}{ccccc} \Gamma(X, \mathcal{F}) & & \xrightarrow{\quad\operatorname{res}_{X_f \subset X}\quad} & & \Gamma(X_f, \mathcal{F}) \\ & \searrow^{\otimes_{\Gamma(X,\mathcal{O}_X)}(\Gamma(X,\mathcal{O}_X)_f)} & & \nearrow_{\cong} & \\ & & \Gamma(X, \mathcal{F})_f & & \end{array}$$

Proof. Cover X with finitely many affine open sets $U_i = \operatorname{Spec} A_i$, let $U_{ij} = U_i \cap U_j$. Cover each U_{ij} with a finite number of affine open sets $U_{ijk} = \operatorname{Spec} A_{ijk}$. Then we have an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

from the definition of sheaf.

Now apply the exact functor $\otimes_{\Gamma(X,\mathcal{O}_X)}\Gamma(X,\mathcal{O}_X)_f$. Since localization commutes with finite products, we have the following exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{O}_X)_f \longrightarrow \underset{\cong \oplus_i\Gamma(\mathrm{Spec}(A_i)_f,\mathcal{F})}{\oplus_i\Gamma(U_i,\mathcal{F})_f} \longrightarrow \underset{\cong \oplus\Gamma(\mathrm{Spec}(A_{ijk})_f,\mathcal{F})}{\oplus\Gamma(U_{ijk},\mathcal{F})_f}$$

In place of $\Gamma(X,\mathcal{O}_X)_f$, $\Gamma(X_f,\mathcal{O}_X)$ makes the above exact sequence hold, thus $\Gamma(X,\mathcal{O}_X)_f \cong \Gamma(X_f,\mathcal{O}_X)$. □

Remark. *Recall that we have the Qcqs lemma [7.3.5, [1]] which says that for X a quasicompact quasiseparated scheme and $s \in \Gamma(X,\mathcal{O}_X)$, then the natural map $\Gamma(X,\mathcal{O}_X)_s \to \Gamma(X_s,\mathcal{O}_X)$ is an isomorphism. In both of the cases, note that how quasicompact and quasiseparated hypothesis were used in an easy way: to obtain finite product which commutes with localization.*

Corollary 1.1 (Important Ex 13.3.F, [1])**.** *Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $\pi_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

Proof. For any affine open $V = \mathrm{Spec}\, B$ of Y, $\pi^{-1}(V)$ is a quasicompact and quasiseparated subscheme of X. Let $f \in B = \Gamma(V,\mathcal{O}_Y)$, and $\pi^{\#}f \in \Gamma(X,\mathcal{O}_X)$ the image of f under the induced map of sheaves, by Proposition 1.2,

$$\Gamma(\pi^{-1}(V),\mathcal{F})_{\pi^{\#}f} \cong \Gamma(\pi^{-1}(V)_{\pi^{\#}f},\mathcal{F}).$$

This is precisely $\Gamma(\mathrm{Spec}\, B,\pi_*\mathcal{F})_f \cong \Gamma(\mathrm{Spec}\, B_f,\pi_*\mathcal{F})$. By 1.1, $\pi_*\mathcal{F}$ is quasicoherent. □

Example 1.1 (Ex 13.3.G, [1])**.** *If A is a ring, and $f \in A$, then the nilradical $\mathfrak{R}(A_f) \cong \mathfrak{R}(A)_f$. So we define the **sheaf of nilpotents** $\mathcal{N}$ on a scheme X, for any affine open, let $\Gamma(\mathrm{Spec}\, A_i,\mathcal{N}) := \mathfrak{R}(A_i)$. This is clearly a quasicoherent sheaf by Proposition 1.1. This is an example of an ideal sheaf (of $\mathcal{O}_X$).*

Proposition 1.3 (Important Ex 13.3.H, [1])**.** *Suppose X is a quasicompact and quasiseparated scheme, $\mathcal{L}$ is an invertible sheaf on X, let $f \in \Gamma(X,\mathcal{L})$ and X_f the open subset of X where f doesn't vanish. Let $\mathcal{F}$ be a quasicoherent sheaf on X. Given any section $t \in \Gamma(X_f,\mathcal{F})$, then for some $n > 0$, the section $f^n t \in \Gamma(X_f,\mathcal{F}\otimes\mathcal{L}^{\otimes n})$ extends to a global section of $\mathcal{F}\otimes\mathcal{L}^{\otimes n}$.*

Proof. We first prove the following claim.

Claim. For X a quasicompact scheme (we don't use quasiseparatedness in the proof of the claim), given a global section $s \in \Gamma(X,\mathcal{F})$ whose restriction to X_f is 0. Then for some $N > 0$, we have $f^N s = 0$ as a global section of $\mathcal{F}\otimes\mathcal{L}^{\otimes N}$.

Proof of Claim. Cover X with a finite number of open affines $U = \mathrm{Spec}\, A$ such that $\mathcal{L}|_U$ is free. Let $\psi : \mathcal{L}|_U \cong \mathcal{O}_X|_U$ be an isomorphism. Since $\mathcal{F}$ is quasicoherent, by definition, there is an A-module M with $\mathcal{F}_U \cong \widetilde{M}$. The section $s \in \Gamma(X,\mathcal{F})$ restricts to give an element $s \in M$. On the other hand, the section $f \in \Gamma(X,\mathcal{L})$ restricts to give a section of $\mathcal{L}|_U$ which give s a section $g = \psi(f) \in A$. Clearly, $X_f \cap U = D(g)$. Since $s|_{X_f}$ is zero, $g^n s = 0$ for some $n > 0$. From the isomorphism

$$\mathrm{id}\times\psi^{\otimes n} : \mathcal{F}\otimes\mathcal{L}^n|_U \cong \mathcal{F}|_U,$$

we conclude that $f^n s \in \Gamma(U, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is zero. Since there are finitely many U, we can take one N large enough to work for each U in the finite cover, thus we have $f^N s = 0$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes N})$ □

Take a finite affine cover $X = \bigcap_i U_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$, given $t \in \Gamma(X_f, \mathcal{F})$, we restrict t to each $X_f \cap U_i$ to get an element of $\mathcal{F}(X_f \cap U_i) = (M_i)_f$. Then by definition of localization, there is an element $t_i \in M_i$ and $n_i > 0$ such that $t_i = f^{n_i} t$. Since there are finitely many i, take one n large enough such that $f^n t = t_i$ work for each $X_f \in U_i$. Now we have sections $t_i \in \Gamma(U_i, \mathcal{F})$ such that on each $(U_i \cap U_j)_f = X_f \cap U_i \cap U_j$, they are both equal to $f^n t$.

Since X is quasiseparated, $U_i \cap U_j$ is quasicompact, applying the claim, there exists $m_{ij} > 0$ such that $f^{m_{ij}}(t_i - t_j) = 0$ on $U_i \cap U_j$. Since there are finitely many m_{ij}, take one m large enough, we have $f^m(t_i - t_j) = 0$ on each $U_i \cap U_j$. Thus these local sections $f^m t_i$ in $\Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ glue together to give a global section $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ whose restriction to X_f is $f^{n+m} t$. □

Example 1.2 (Less Important Ex 13.3.I, [1])**.** *A counterexample that Proposition 1.2 needs not hold without the quasicompact hypothesis.*

Proof. To find such an example is to find an example that localization does not commute with infinite direct product. Take X to be the infinite disjoint union $X = \amalg \operatorname{Spec} \mathbb{Z}$, let $\Gamma(\operatorname{Spec} \mathbb{Z}, \mathcal{F}) = \mathbb{Q}$, $f = (2, 2, \dots) \in \Gamma(X, \mathcal{F})$. Then $(\prod_1^\infty \mathbb{Q})_{(2,2,\dots)} \ncong \prod_1^\infty \mathbb{Q}_2$, since the element $(1, 1/2, 1/2^2, \dots, 1/2^n, \dots)$ is not in the left hand side. □

References

[1] Vakil, R. (2017). *The Rising Sea: Foundations of Algebraic Geometry.* Preprint.

[2] Hartshorne, R. (2013). *Algebraic geometry* (Vol. 52). Springer Science & Business Media.