1 Generic Freeness and Chevalley's Theorem

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A simple example with neither closed nor open image: $\mathbb{A}_k^2 \to \mathbb{A}_k^2, (x, y) \mapsto (x, xy)$ for which the image is the plane with the y-axis removed, but the origin put back in. The following definition of constructible sets can capture this phenomenon.

1.1 Some Properties of Constructible Sets

[Exercises 7.4.A-B,[1]]

Definition 1.1. *A* ***constructible subset*** *of a Noetherian topological X space is an element of Boolean algebra generated by the open subsets of X which is the smallest family of subsets such that*

(i) every open set is in the family; (ii) a finite intersection of family member is in the family; and (iii) the complement of a family member is also in the family.

For example,the image of $(x, y) \mapsto xy$ is constructible.

Recall that a subset of a topological space X is **locally closed** if it is an intersection of an open and a closed subset.

Proposition 1.1. *A subset of a Noetherian topological space is constructible if and only if it is the finite* ***disjoint*** *union of locally closed subsets.*

Proof. By definition, a constructible subset is a finite union of locally closed subsets. Let $C \subset X$ a constructible subset of a Noetherian topological space X, write

$$C = (U_1 \cap K_1) \cup (U_2 \cap K_2) \cup \cdots \cup (U_s \cap K_s)$$

where each U_i is open and K_i is closed, and each $(U_i \cap K_i)$ is not contained in any other. Assume $(U_i \cap K_i) \cap (U_k \cap K_j) \neq \emptyset$, then since

$$U_j \cap K_j \backslash U_i \cap K_i = [U_j \cap (K_j \backslash U_i)] \cup [(U_j \backslash K_i) \cap K_j],$$

we can refine the decomposition by replacing $U_i \cap K_i$ by $[U_j \cap (K_j \backslash U_i)] \cup [(U_j \backslash K_i) \cap K_j]$ where the two locally closed subsets in the union are properly contained in $U_i \cap K_i$ and discard redundant terms. Noetherianess of the topological space X guarantees that this process stops, thus we get a decomposition into disjoint union of locally closed sets. □

As a consequence, if $X \to Y$ is a continuous map of Noetherian topological spaces, then the preimage of a constructible set is constructible. (The hypothesis that X, Y are Noetherian is because constructible sets are only defined for Noetherian topological spaces for the moment.)

Proposition 1.2. *The generica point of $\mathbb{A}_k^1$ does not form a constructible subset of $\mathbb{A}_k^1$, where k is a field.*

Proof. If the generic point (0) is constructible then itself is locally closed. If (0) is contained in some closed set $V(I)$ then $V(I)$ must be $\operatorname{spec} k[x]$. But $\{(0)\}$ is not open. Indeed, the $\operatorname{Spec} k[x] \setminus \{(0)\}$ consists of all closed points (all nonzero prime ideals are maximal). Moreover, by a similar argument to Euclid's proof of the infinitude of primes of $\mathbb{Z}$, one can show that there are infinitely many closed points. Thus (0) is not constructible. □

Proposition 1.3. *(a) A constructible subset of a Noetherian scheme is closed if and only if it is "stable under specialization". More precisely, if Z is a constructible subset of a Noetherian scheme X, then Z is closed if and only if for every pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$, if $y_1 \in Z$ then $y_2 \in Z$;*

(b) A constructible subset is open if and only if it is "stable under generalization".

Proof. (a) The "only if" direction is clear. For the "if" direction, by decompose a closed subset of X into a finite union of irreducible closes subsets [Prop 3.6.15, [1]], we may write $Z = \bigcup_{i=1}^{n} U_i \cap Z_i$ where U_i are open and Z_i are closed and irreducible. Now take a generic point z_i for each Z_i, then $\overline{\{z_i\}} = Z_i \subset Z$ by the condition, thus $Z = \bigcup Z_i$.

(b) Since a constructible subset is open if and only if its complement is a closed constructible subset, (b) follows from (a).

□

For any subset S of a scheme Y, S can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $\pi : X \to Y$ be the natural map.

Theorem 1.1 (Chevalley's Theorem)**.** *If $\pi : X \to Y$ is a finite type morphism of Noetherian schemes, the image fo any constructible set is constructible. In particular, the image of π is constructible.*

In order to prove Chevalley's theorem, we show the following lemma following a sequence of exercises (Exercises 7.4.F-7.4.K, [1]).

1.2 Grothendieck's Generic Freeness Lemma

We say *a B-algebra A satisfies (†) if for each finitely generated A-module M, there exists a nonzero $f \in B$ such that M_f is a free B_f-module.*

Proposition 1.4 (Grothendieck's Generic Freeness Lemma)**.** *Suppose B is a Noetherian integral domain. Then every finitely generated B-algebra satidfies (†).*

Lemma 1.1 (Ex 7.4.K, [1])**.** *Suppose M is a B-module that is an increasing union of submodules M_i, with $M_0 = 0$, and every M_{i+1}/M_i is free, then M is free.*

Proof. We construct compatible isomorphisms $\phi_n : \oplus_{i=0}^{n-1} M_{i+1}/M_i \to M_n$ by induction on n. The base case is trivial: $M_1/M_0 = M_1$. For the induction step, since M_i/M_{i-1} is free, the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

splits. Thus there exists an isomorphism $M_{i-1} \oplus M_i/M_{i-1} \cong M_i$. Applying the isomorphism ϕ_{i-1}, we obtain a compatible isomorphism ϕ_i. Since colimit is a functor $\lim_I$: $\mathrm{Fun}(I, B\text{-Mod}) \to B\text{-Mod}$, thus it takes isomorphism to isomorphism. So $\phi := \varinjlim \phi_n$: $\oplus_{i=0}^{\infty} M_{i+1}/M_i \to M$ is an isomorphism, thus M is free. □

Remark. *The same argument applies to the case that if M_i/M_{i-1} are all projective modules, then M is isomorphic to their direct sum. Note that this is the induction step of Theorem A.1.*

Proof of Proposition 1.4. Let B be an integral domain. We first show that B itself satisfies (†) [Ex 7.4.F, [1]]. Let M be a finitely generated B-module and $x_1, x_2, \dots x_n \in M$ a minimal number of generators. Let $M_1 = x_1 B \subset M$, then $M_1 \cong B/I_1$ where $I_1 = \{b \in B | bx_1 = 0\}$. Note that M/M_1 has $n-1$ generators, thus by induction, we can show that there exists a filtration by B-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong B/I_i$ for some ideal I_i. (Furthermore, one can show that each I_i is indeed prime.) Now take $f \neq 0$ contained in every nontrivial I_i, since B is an integral domain, such nonzero f exists ($\prod I_i$ for nontrivial I_i's is nonzero.) Then $(M_i/M_{i-1})_f$ is either 0 or isomorphic to B_f, so $M_f \cong \oplus (M_i/M_{i-1})_f$ is a free B_f module.

Since every finitely generated B-algebra A can be written as $A = B[x_1, x_2, \dots x_n]/I$ and every A-module can be naturally viewed as a $B[x_1, x_2, \dots x_n]$-module M such that $IM = 0$. We reduce the case to free B-algebra $B[x_1, x_2, \dots x_n]$.

By induction, we reduce to prove the following: if A is a finitely generated B-algebra satisfying (†), then $A[T]$ does too [Ex 7.4.G].

We now prove this statement. Suppose A satisfies (†), and let M be a finitely generated $A[T]$-module, generated by the finite set S. Let $M_0 = 0$, and let M_1 be the sub-module of M generated by S. For $n > 0$, inductively define

$$M_{n+1} = M_n + TM_n,$$

a submodule of M. Note that M is the increasing union of the A-modules M_n. Define $F_i = \{x \in M_1 | T^i f \in M_i\}$, then we have $M_{i+1}/M_i \cong M_1/F_i$ and $F_1 \subset\subset F_2 \subset \cdots \subset F_i \subset \dots$ is an ascending chain of A-submodules of M_1 which must terminates since M_1 is Noetherian. Let $F_0 = 0$, from what we have proved at the beginning and induction hypothesis on A, each M_1/F_i as and A-module satisfies (†). So there is a filtration

$$0 \subset G_{i,0} \subset G_{i,1} \subset \cdots \subset G_{i,r} \subset M_1/F_i = M_{i+1}/M_i$$

such that $G_{i,k}/G_{i,k-1} \cong B/J_{i,k}$ for some prime ideal $J_{i,k}$ of B. Now take a nonzero element $g \neq 0$ contained in every nontrivial $J_{i,k}$, then each $(M_{i+1}/M_i)_g$ is a free B_g-module, thus $M_g \cong (M_{i+1}/M_i)_g$ is a free B_g-module by lemma 1.1. (If you think you have missed anything, look at the proof at the beginning which is the base case of our induction.) Now we have finished the proof. □

1.3 Proof of Chevalley's Theorem

Chevalley's Theorem(Theorem 1.1) If $\pi : X \to Y$ is a finite type morphism of Noetherian schemes, the image of any constructible set is constructible. In particular, the image of π is constructible.

Proof. First we can reduce to the case where both X and Y are affine: indeed there are finite affine covers $\{U_i\}$ of X, $\{V_j\}$ of Y such that $f(U_i) \subset V_i$, Let $f_i := f|_{U_i} : U_i \to V_i$. Then for every $S \subset X$ constructible, $f(S) = \bigcap_i f_i(S \cap U_i)$. So if each $f_i(S \cap U_i)$ is constructible, $f(S)$ is constructible.

For A Noetherian, every open subset of spec(A) is a finite union of affine schemes and every closed subset $V(I)$ is homeomorphic to Spec A/I. Furthermore, we can deal with irreducible components of Y separately (there are finitely many irreducible components since Y is Noetherian), so we may assume Y is irreducible.

So we are left to prove the following:

Lemma 1.2. *If $\pi : X \to Y$ is a finite type morphism of Noetherian affine schemes with Y irreducible,then the image of π is constructible.*

Proof. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, then by the Generic Freeness Lemma 1.4, there is a nonzero $f \in B$ such that A_f is a free B_f-module. It must have zero or positive rank. Note that $\pi^{-1}(D(f)) = \operatorname{Spec} A_f$ and the morphism $\pi|_{\pi^{-1}(D(f))} : \pi^{-1}(D(f)) \to D(f)$ is determined by $B_f \to A_f$. Thus if A_f is of rank 0, then $\pi^{-1}(D(f)) = \emptyset$. If A_f is of positive rank n, then $A_f \cong \oplus_n B_f$, $\pi^{-1}(D(f)) = \bigsqcup_n \operatorname{Spec} B_f = \bigsqcup_n D(f) \supseteq D(f)$. Let $U = D(f)$ which is dense and constructible, thus we have shown that there is a dense open subset $U \subset Y$ such that the image of π either contains U or else do not meet U. If $U = Y$, then we are done. If not, consider $Y \setminus U$ which is a closed subset homeomorphic to some $\operatorname{Spec} B/I$. So we can repeat the argument. This process involves only finitely many steps since Y is Noetherian. Thus we have shown that the image of π is a finite union of constructible subsets (dense subsets) hence constructible. □

□

1.4 Applications of Chevalley's Theorem

Proposition 1.5 (Cor 2.2.11, [3])**.** *Let k be a field and X a scheme of finite type over k. For $x \in X$, x is closed if and only if the residue field $\mathbb{K}(x)$ of x is an algebraic extension of k.*

Proof. Suppose $x \in X$ is closed and let $U = \operatorname{Spec} R$ be an affine neighbourhood of x. Then x is closed in U and hence $\{x\}$ is a constructible subset of U. Since U is of finite type, we write $R \cong k[x_1, \dots x_n]/I$. Let $p_i : U \to \mathbb{A}^1_k$ be the morphism induced by the ring homomorphism $k[x] \to k[x_1, \dots x_n]/I, x \mapsto x_i$, which is clearly of finite type. So by Chevalley's Theorem, $p_i(x)$ is a constructible point of $\mathbb{A}^1_k$. Now by Proposition 1.2, the generic point of $\mathbb{A}^1_k$ is not constructible. Thus $k(p_i(x))$ is algebraic over k. Since the residue field $\mathbb{K}(x)$ is generated by the subfield $k(p_i(x))$, $\mathbb{K}(x)$ is algebraic over k.

Conversely, let $x = (p) \in U = \operatorname{Spec} R$. Since $\mathbb{K}(x) \supset R/p \supset k$ and $\mathbb{K}(x)$ is algebraic over k, R/p is also algebraic. Note that if a is algebraic over k, then $k[a]$ is already a field. Thus R/p is a field, p is a closed point. □

Indeed, the "only if" direction above has already shown the following, but we rewrite the proof in a neater way.

Theorem 1.2. ***[Hilbert's Nullstellensatz]*** *If k is any field, every maximal ideal of $k[x_1, \dots x_n]$ has residue field a finite extension of k. In other words, any field extension of k that is finitely generated as a ring is necessarily also finitely generated as a module (i.e., is a finite extension of fields).*

Proof. We want to show that if K is a field extension of k that is finitely generated as a k-algebra, say by $x_1, \dots x_n$, then it is a finite extension of fields. It suffices to prove that each x_i is algebraic over k. But if x_i is not algebraic over k, then we have an inclusion of rings $k[x_i] \to K$, corresponding to a dominant morphism $\pi : \operatorname{Spec} K \to \mathbb{A}^1_k$ of finite type k-scheme. The image of π is one point since $\operatorname{Spec} K$ is. By Chevalley's Theorem and Proposition 1.2, the image of π is not the generic point of $\mathbb{A}^1_k$, so $\operatorname{im}(\pi)$ is a closed point of $\mathbb{A}^1_k$ thus π is not dominant, a contradiction. (A rational map of irreducible schemes is dominant if and only if π sends the generic point of X to the generic point of Y, see Exercise 6.5.A, [1].) □

Corollary 1.1. ***[Hilbert's Weak Nullstellensatz]*** *If k is an algebraically closed field, then the maximal ideals of $k[x_1, \dots x_n]$ are precisely those ideals of the form $(x_1 - a_1, \dots x_n - a_n)$, where $a_i \in k$.*

Proof. If $\mathfrak{m}$ is a maximal ideal of R then $R/\mathfrak{m}$ is a field which is finitely generated as a k-algebra. By the previous theorem it is an algebraic extension of k, hence equal to k. Therefore each x_i maps to some $ai \in k$ under the natural map $R \to R/\mathfrak{m} = k$, so $\mathfrak{m}$ contains the ideal $(x_1 - a_1, \dots, x_n - a_n)$. This is a maximal ideal, so it equals $\mathfrak{m}$. □

As a consequence, a family of polynomial functions on k^n with no common zeros generates the unit ideal of R. This is the Nullstellensatz one would see in an elementary algebraic geometry course.

A similar idea of proof of Hilbert's Nullstellensatz can be used to prove the following consequences:

Proposition 1.6. *Suppose* $\pi : X \to \operatorname{Spec} k$ *is a quasifinite morphism. Then* π *is a finite.*

An affine scheme that is reduced and of finite type over k is said to be an **affine variety (over** k**)**, or an **affine** k**-variety** [Definition 5.3.7, [1]].

Proposition 1.7. *A morphism of affine* k*-varieties* $\pi : X \to Y$ *is surjective if and only if it is surjective on closed points (i.e. if every closed point of* Y *is the image of a closed point of* X*.)*

A Appendix

Here we show a nice result which is a further generalisation of Lemma 1.1.

Theorem A.1 (Auslander)**.** *[Theorem 5.22, [2]] Let M be a right R-module, I a nonempty well-ordered set and $\{M_i\}_{i\in I}$ a family of submodules such that:*

1. $M_i \subset M_j$ if $i \leq j$;

2. $M = \bigcup_{i\in I} M_i$;

3. $\mathrm{pd}(M_i/M_i') \leq n$ where $M_i' = \bigcap_{j<i} M_j$;

then $\mathrm{pd}\, M \leq n$, where pd denotes the projective dimension.

Proof. By induction on n. The case $n = 0$ uses the same argument as in the proof and remark of Lemma 1.1.

Now assume the result for $n-1$. We are given that $\mathrm{pd}\, M_i/M_i' \leq n$ for all $i \in I$. Let F be the free module with free basis M; F_i be the free module with free basis M_i; F_i' be the free module with free basis M_i'. Let K be the kernel of the map $F \to M$ and we have a short exact sequence $0 \to K \to F \to M \to 0$. Define $K_i = F_i \cap K$ and $K_i' = F_i' \cap K$, then we have a short exact sequence $0 \to K_i \to F_i \to M_i \to 0$. The kernel of the map $F_i/F_i' \to M_i/M_i'$ is $K_i/K_i \cap F_i' = K_i/F_i \cap K \cap F_i' = K_i/K \cap F_i' = K_i/K_i'$, thus we have a short exact sequence

$$0 \to K_i/K_i' \to F_i/F_i' \to M_i/M_i' \to 0.$$

Each F_i/F_i' is free since F_i has a set of generators, a subset of which generates F_i'. Hence F_i/F_i' is projective.

By generalised Schanuel's Lemma, we have the following: Suppose that $0 \to K \to P \to A \to 0$ is an exact sequence with P projective and A <u>not</u> projective. Then $\mathrm{pd}\, K < \infty$ if and only if $\mathrm{pd}\, A < \infty$ and we have in this case $1 + \mathrm{pd}\, K = \mathrm{pd}\, A$ [Lemma 5.20, [2]].

So by the above, $\mathrm{pd}\, K_i/K_i' \leq n-1$. It can be checked that:

1. $\forall i < j$, $i, j \in I$, $K_i \subset K_j$;

2. $K = \cap_{i\in I} K_i$ and $K_i' = \cap_{j<i} K_j$. That is, the induction hypothesis apply to K and the family $\{K_i\}$. So $\mathrm{pd}\, M \leq 1 + \mathrm{pd}\, K \leq n$. □

References

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