Applications of Krull’s Principal Ideal Theorem

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This post is about some applications of Krull’s Principal Ideal Theorem and regular local rings in dimension theory and regularity of schemes [Part IV, Vakil], with the aim of connecting the 2018-2019 Warwick course MA4H8 Ring Theory with algebraic geometry. The lecture notes/algebraic references are here: 2018-2019 Ring Theory. Note that the algebraic results included here follow those in [Vakil, [1]] either as exercises or proved results for which I have included the references.

Besides including results in both their geometric and algebraic statements, I have given proofs to a selection of exercises in Part IV, [Vakil, [1]] to illustrate more applications and other connections to the contents in the Ring Theory courses. The indexes for exercises follow those in [Vakil, [1]].

1 Krull’s Principal Ideal Theorem

Krull’s Principal Ideal Theorem (algebraic version). [P.23, [2]] Let $R$ be a Noetherian ring. Let $a \in R$ be a non unit, suppose that $P$ is a prime ideal minimal over $a$. Then $\text{rk} \ P \leq 1$. If furthermore, $a$ is not a zero divisor, then $\text{rk} \ P = 1$.

Krull’s Principal Ideal Theorem (geometric version). [P. 316, [1]] Suppose $X$ is a locally Noetherian scheme, and $f$ is a function. The irreducible components of $V(f)$ are codimension 0 or 1.

See [P.23, [2]] for the proofs.

The Generalised Principal Ideal Theorem. Let $R$ be a commutative Noetherian Ring. Suppose that $P$ is a prime ideal minimal over the elements $x_1, \ldots, x_r \in R$, then $\text{rk} \ P \leq r$.

Proof. We prove by induction. For $r = 1$, this is Krull’s Principal Ideal Theorem.

Assume the result is true for primes minimal over $\leq r-1$ elements. Suppose $P$ is minimal over $x_1, \ldots, x_r$ and suppose there exists a chain of primes $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{r+1}$. If $x_1 \in P_k$ then since $P_0/x_1R$ is minimal over $\bar{x}_2, \ldots, x_r$ in the ring $R/x_1R$ we get a contradiction.

Let $k$ be such that $x_1 \in P_k$ but $x_1 \notin P_{k+1}$. So we have $P_k \supseteq P_{k+2} + x_1R \supseteq P_{k+2}$ hence a chain $P_k/P_{k+2} \supseteq x_1 + P_{k+2} \supseteq P_{k+2}/P_{k+2}$. Since $P_k/P_{k+2} \supseteq P_{k+1}/P_{k+2} \supseteq P_{k+2}/P_{k+2}$, by Krull’s Principal Ideal Theorem, $P_k/P_{k+2}$ cannot be minimal over $x_1 + P_{k+2}$. So there exists a prime ideal $P'_{k+1}$ such that $P_k \supseteq P'_{k+1} \supseteq x_1 + P_{k+2} \supseteq P_{k+2}$. Proceeding this way, we can build a new chain $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_k \supseteq P'_{k+1} \supseteq \cdots P'_{r} \supseteq P_{r+1}$ with $x_1 \in P'_{r}$ which leads to a contradiction as before.

Exercise 11.3.A Show that an irreducible homogeneous polynomial in $n+1$ variables over a field $k$ describes an integral scheme of dimension $n - 1$.

Proof. Let $f(x_0, \ldots, x_n)$ be the homogeneous equation cut out the scheme in question. Note that it is covered by spec $k[x_0/i_0, \ldots, x_l/i_l, \ldots, x_n/i_n]/(f(x_0/i, \ldots, x_l/i_l, \ldots, x_n/i_n))$ and $f$ is irreducible, then apply Krull’s Principal Ideal Theorem.

Exercise 11.3.B. [Lemma 4.38,[2]] Suppose $(R, m)$ is a Noetherian local ring, and $f \in m$. Then $\dim R/(f) \geq \dim R - 1$. 
Proof. Let \( n = \dim A \), then \( \text{rk} \, \mathfrak{m} = n \), so there exists a chain of primes \( \mathfrak{m} = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \). As in the proof of the Generalised Principal Ideal Theorem, we can construct a new chain of primes \( \mathfrak{m} = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_{n-1} \) with \( f \in Q_{n-1} \). Hence, \( \text{rk}(\mathfrak{m}/(f)) \geq n - 1 \).

We can further more show that if \( f \) is regular (nonzerodivisor) then the equality holds. If \( \mathfrak{m}/cR = T_0/(f) \supseteq \cdots \supseteq T_k/(f) \) is a chain of primes in \( R/(f) \) then \( J = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k \) is a chain of primes in \( R \). Since \( f \) is regular, \( T_k \) is not a minimal prime of \( R \). Thus, the regular elements of \( R \) is the complement of the union of all minimal primes, see Proposition 4.18, \([2]\). So \( n = \text{rk} \, \mathfrak{m} = \text{rk} \, \mathfrak{m}/(f) + 1 \). Hence, \( \text{rk} \, \mathfrak{m}/(f) = n - 1 \). \qed

**Important Exercise 11.3.C** (a) (Hypersurfaces meet everything of dimension at least 1 in projective space, unlike in affine space.) Suppose \( X \) is a closed subset of \( \mathbb{P}^k_n \) of dimension at least 1, and \( H \) is a nonempty hypersurface in \( \mathbb{P}^k_n \). Show that \( H \) meets \( X \).

(b) Suppose \( X \hookrightarrow \mathbb{P}^k_n \) is a closed subset of dimension \( r \). Show that any codimension \( r \) linear space meets \( X \).

(c) Show further that there is an intersection of \( r + 1 \) nonempty hypersurfaces missing \( X \). If \( k \) is infinite, show that there is a codimension \( r + 1 \) linear subspace missing \( X \).

(d) If \( k \) is an infinite field, show that there is an intersection of \( r \) hyperplanes meeting \( X \) in a finite number of points.

**Proof.** (a) Consider the affine cone \( CX \subseteq A^{n+1}_k \) over \( X \). Let \( \dim X = n \), then \( \dim CX = n + 1 \), by Krull’s Principal Ideal Theorem, \( \dim CX \cap V(f) \geq n + 1 - 1 = n \geq 1 \). Note that \( \dim CX \cap V(f) \) contains the origin. Since it has dimension \( \geq 1 \), it also contains another point in \( A^{n+1}_k \setminus \{0\} \hookrightarrow \mathbb{P}^n_k \).

(b) Similar to (a), note that \( \dim CX \cap V(f_1, \ldots, f_n) \geq \dim X + 1 - r \geq 1 \).

(c) The key step: show that there is a hypersurface of sufficiently high degree that doesn’t contain any generic point of \( X \). Show this by induction on the number of generic points. To get from \( m \) to \( m + 1 \): take a hypersurface not vanishing on \( p_1, \ldots, p_m \). If it doesn’t vanish on \( p_{m+1} \), we are done. Otherwise, call this hypersurface \( f_{m+1} \). Do something similar with \( m + 1 \) replaced by \( i \) for each \( 1 \leq i \leq m \). Then consider \( \Sigma_i f_1 \cdots f_{i} \cdots f_{m+1} \), this hypersurface doesn’t vanish at any \( p_m \).

Let \( X_{-1} = X \) and \( g_{-1} = 0 \), take \( g_i \) to be a hypersurface that doesn’t contain any generic point of \( X_i = X_{i-1} \cap V(g_{i-1}) \). Then by Krull’s principal Ideal Theorem, \( \dim X_i = \dim X_{i-1} - 1 \). So we have \( \dim X_{r+1} = \dim X - (r + 1) = -1 \), thus the intersection of the \( r + 1 \) hypersurfaces \( g_i \) and \( X \) is empty.

If \( k \) is finite, we can choose a closed point \( [a_0 : \ldots : a_n] \) for each irreducible component and choose a homogeneous linear form that does not vanish at any of these points.

(d) As in (c), we can take \( r \) linear forms \( g_i \) such that \( \dim X_i = 0 \) which means the intersection of \( X \) with the \( r \) hyperplanes consists of finitely many points. Note that a Noetherian ring with dimension 0 has only finitely many primes. \qed

2 Regularity and Smoothness

2.1 Regular Local Ring

\([4.7, \[2]\] \) Let \( R \) be a Noetherian local ring with Jacobson radical \( (\text{the maximal ideal}) \) \( J \). We have \( V(R) := \dim J/J^2 \) as a vector space over the field \( R/J \). So \( V(R) = \) the number of elements in a minimal generator set for \( J \). By the Generalised Principal Ideal Theorem, we have \( \text{rk} \, J \leq V(R) \) [also see Theorem 12.2.1, \([1]\)].
(Fact(Cor 2.23,[2]): $x_1, \ldots, x_k$ is a minimal generating set for $J \iff \bar{x}_1 \ldots \bar{x}_k$ is a basis for the vector space $J/J^2$ over $R/J$.)

**Definition 2.1.** A Noetherian local ring is called a regular local ring if $\text{rk } J = V(R)$.

**Important Exercise 12.1.B.** [Krull’s Principal Ideal Theorem for Tangent Spaces]. Suppose $A$ is a ring, and $m$ a maximal ideal. If $f \in m$, show that the Zariski tangent space of $A/(f)$ is cut out in the Zariski tangent space of $A$ by $f(\mod m^2)$. Hence the dimension of the Zariski tangent space of $\text{Spec } A/(f)$ at $[m]$ is the dimension of the Zariski tangent space of $\text{Spec } A$ at $[m]$, or one less.

Note that localization and quotient commutes, we can translate the above into the following algebraic statement.

**Algebraic translation**[Lemma 4.40, [2]]. Let $R$ be a Noetherian local ring with Jacobson radical $J$ ($R$ not a field). Suppose that $x \in J \setminus J^2$, let $R^* = R/xR$. Then $V(R^*) = V(R) - 1$. (If $x \in J^2$, then $V(R^*) = V(R)$.)

**Proof.** Note that $R^*$ is a Noetherian local ring with Jacobson radical $J^* = J/xR$. Let $y_1^*, \ldots, y_k^*$ be a minimal generating set for $J^*$. Choose $y_1, \ldots, y_k \in J$ such that $y_i \mapsto y_i^*$ under the natural homomorphism $R \to R/xR$. Claim $x, y_1, \ldots, y_k$ is a minimal generating set for $J$ which is equivalent to that $\bar{x}, \bar{y}_1, \ldots, \bar{y}_k$ is a basis for the vector space $J/J^2$ over $R/J$. We shall now show that $\bar{x}, \bar{y}_1, \ldots, \bar{y}_k$ in the vector space $J/J^2$ are linearly independent.

Suppose that $xr + y_1r_1 + \cdots + y_kr_k \in J^2$. So $y_1^*r_1^* + \cdots + y_k^*r_k^* \in (J^*)^2$ where $r_i^*$ are the homomorphic images of $r_i$ under $R \to R/xR$. Since $y_1^*, \ldots, y_k^*$ is a minimal generating set for $J^*$, $\bar{y}_1^*, \ldots, \bar{y}_k^*$ is a basis for the vector space $J^*/(J^*)^2$ over $R/J^*$. It follows that $r_i^* \in J^*$, so $r_i \in J$. Then $xr \in J^2$. If $r \notin J$ then $r$ is a unit, it follows that $x \in J^2$, a contradiction. Thus $r \in J$. This proves the claim. (If $x \in J^2$, from the above we can easily see that the claim becomes that $\bar{y}_1, \ldots, \bar{y}_k$ is a basis for the vector space $J/J^2$.)

We now look at some applications of this results:

**Exercise 12.1.D** Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a codimension 1 ideal that is not principal, using the method of Solution 12.1.4. (See Figure 12.2 for the projectivization of this situation — a line on a smooth quadric surface.)

![Figure 12.2. The line $V(x, z)$ on the smooth quadric surface $V(wz - xy) \subset \mathbb{P}^3$.](image-url)
Proof. Let \( A = k[w, x, y, z]/(wz - xy) \), then \( \dim A = 3 \) by Krull’s Principal Ideal Theorem. As \( A/(x, y) \cong k[w, y] \), \( \dim A/(x, y) = 2 \), so \( (x, z) \) has codimension 1. And \( \text{Spec } A/(x, y) \) has Zariski tangent space of dimension 2 at the origin. But \( \text{Spec } A/(f) \) must have Zariski tangent space of dimension at least 3 by Exercise 12.1.B.

Exercise 12.1.E Let \( A = k[w, x, y, z]/(wz - xy) \). Show that \( \text{Spec } A \) is not factorial.

Proof. This follows from Exercise 12.1.D and that all codimension 1 prime ideal in a unique factorisation domain is principal (Lemma 11.1.6, [I]).

Remark. A related algebraic result is Proposition 7.5, [2]: Let \( R \) be a commutative Noetherian integral domain. Then \( R \) is a UFD if and only if every rank 1 prime ideal of \( R \) is principal.

Remark (Vakil). As \( A \) is integrally closed if \( \text{char } k \neq 2 \) (Exercise 5.4.I(c), [I]), this yields an example of a scheme that is normal but not factorial.

Exercise 12.2.B Suppose \( X \) is a finite type \( k \)-scheme (such as a variety) of pure dimension \( n \), and \( p \) is a nonsingular closed point of \( X \), so \( \mathcal{O}_{X, p} \) is a regular local ring of dimension \( n \). Suppose \( f \in \mathcal{O}_{X, p} \). Show that \( \mathcal{O}_{X, p}/(f) \) is a regular local ring of dimension \( n - 1 \) if and only if \( f \in \mathfrak{m} \setminus \mathfrak{m}^2 \).

Proof. In the algebraic translation of Important Exercise [12.1.B] we have that \( V(R^*) = V(R) - 1 \) if \( x \in J \setminus J^2 \); and \( V(R^*) = V(R) \) if \( x \in J^2 \). Now we continue this algebraic statement, we show that for \( x \in J/J^2 \),

\[
V(R) - 1 = V(R^*) \geq \text{rk } J^* \geq \text{rk } J - 1 \quad \text{(by Exercise 11.3.B)} = V(R) - 1.
\]

So \( V(R^*) = \text{rk } J^* \) and \( R^* \) is regular local.

If \( x \in J^2 \), in the case of the exercise 12.2.B (\( X \) is a finite type \( k \)-scheme), \( x \) is not a zero divisor, so we still have \( \text{rk } J^* = \text{rk } J - 1 \). Since \( V(R) = V(R^*) = \text{rk } J \), we see that in this case \( R^* \) is not regular local. This proves the exercise.

Recall the definition of Cartier divisor [8.4.1,I]: If \( X \to Y \) is a closed embedding, and there is a cover of \( Y \) by affine open subsets \( \text{Spec } A_i \subset Y \), and there exist non-zerodivisors \( t_i \in A_i \) with \( V(t_i) = X_i|_{\text{Spec } A_i} \) (scheme-theoretically—i.e., the ideal sheaf of \( X \) over \( \text{Spec } A_i \) is generated by \( t_i \)), then we say that \( X \) is an effective Cartier divisor on \( Y \).

Exercise 12.2.C [The Slicing Criterion for Regularity] Suppose \( X \) is a finite type \( k \)-scheme (such as a variety), \( D \) is an effective Cartier divisor on \( X \) (Definition 8.4.1), and \( p \in D \). Show that if \( p \) is a regular point of \( D \) then \( p \) is a regular point of \( X \).

Proof. The algebraic translation is that if \( R/(f) \) is regular local for \( f \) regular (a nonzerodivisor), then \( R \) is also regular local. We proceed with a similar argument as the last exercise.

\[
V(R) - 1 \leq V(R^*) \quad \text{(by Important Ex 12.1.B)} = \text{rk } J^* = \text{rk } J - 1 \leq V(R) - 1.
\]

So \( \text{rk } J = V(R) \) and \( R \) is regular.
2.2 Regular local rings are integral domains

Proposition 2.1 (Lemma 4.19, [2]). Let \( R \) be a commutative ring. Let \( P_1, \ldots, P_n \) be ideals of \( R \), at least \( n - 2 \) of which are prime. Let \( S \) be a subring of \( R \). Suppose that \( S \subseteq \bigcap_{i=1}^n P_i \), then \( S \subseteq P_k \) for some \( k, 1 \leq k \leq n \). (Note that \( S \) does not necessarily contain 1 since we do not assume \( R \) is unital.)

**Proof.** Prove by induction on \( n \). For \( n = 1 \), the result is trivial. For \( n = 2 \) if \( S \not\subseteq P_1 \) and \( S \not\subseteq P_2 \) then choose \( x_1, x_2 \in S \) such that \( x_1 \not\in P_2 \) and \( x_2 \not\in P_1 \). Then \( x_1 + x_2 \not\in P_i, i = 1, 2 \).

Now assume \( n > 2 \) and that the result holds for values \( < n \). Clearly any selection of \( n - 1 \) of the \( P_i \) at most 2 will be non-prime. Suppose that \( S \subseteq \bigcap_{i=1}^n P_i \) but \( S \not\subseteq P_i \) for any \( 1 \leq i \leq n \). Then \( S \not\subseteq \bigcap_{i=1,i\neq k} P_i \). Thus \( x_k \in P_k \). Since \( n > 2 \), at least one of the \( P_i \) must be prime, say \( P_1 \). Let \( y = x_1 + x_2 \cdots x_n \), then \( y \not\in P_i \) for any \( 1 \leq i \leq n \), a contradiction. This completes the induction. \( \square \)

Corollary 2.1 (Prime Avoidance, Prop 11.2.13, [1]). Suppose \( p_1, \ldots, p_n \) are prime ideals of a ring \( A \), and \( I \) is another ideal of \( A \) not contained in any \( p_i \). Then \( I \) is not contained in \( \cap p_i \); there is an element \( f \in I \) not in any of the \( p_i \). \( \square \)

Lemma 2.1 (Lemma 4.42, [2]). Let \( R \) be a Noetherian local ring which is not an integral domain. Let \( P = pR(p \in P) \) be a prime ideal. Then \( \text{rk} P = 0 \).

**Proof.** Suppose that \( Q \subseteq P \) where \( Q \) is a prime ideal, then \( p \notin Q \). Now \( q \in Q \) implies \( q = pt \) for some \( t \in R \). Hence \( pt \in Q \Rightarrow t \in Q \) since \( p \notin Q \). So \( q \in pQ \subseteq P^2 \subseteq p^2R \). Preceding this way we have \( Q \subseteq P^n \) for all \( n \geq 1 \), so \( Q \subseteq \bigcap_{n=1}^\infty P^n \subseteq \bigcap_{n=1}^\infty J^n \) where \( J \) is the Jacobson radical of \( R \). But by Theorem 4.9, [2] \( \bigcap_{n=1}^\infty J^n = 0 \), so \( Q = 0 \) which is a contradiction since \( R \) is not a domain. Hence \( \text{rk} P = 0 \). \( \square \)

Theorem 2.1 (Theorem 4.43, [2]/Theorem 12.1.13, [1]). A regular local ring is an integral domain.

**Proof.** By induction on \( \text{dim} R = \text{rk} J \). If \( \text{rk} J = 0 \) then \( R \) must be a field.

Suppose now that \( \text{rk} J = n > 0 \) and assume result for rings of \( \text{dim} < n \). Since \( V(R) = \dim J/J^2 = \text{rk} J \neq 0, J \neq J^2 \). Choose \( x \in J \setminus J^2 \). By Exercise 12.2.B or Theorem 4.41, [2], \( R^x = R/xR \) is regular local. Also \( \text{dim} R^x = \text{dim} R - 1 \). By induction hypothesis, \( R^x \) is an integral domain, that is \( xR \) is a prime ideal. Suppose that \( R \) is not an integral domain, then by Lemma 2.1 \( xR \) is a minimal prime. Let \( P_1, \ldots, P_k \) be the minimal primes of \( R \), then \( x \in P_1 \cap \cdots \cap P_k \), thus \( J \setminus J^2 \subseteq P_1 \cap \cdots \cap P_k \). \( J \subseteq J^2 \cap P_1 \cap \cdots \cap P_k \). So \( J \subseteq P_j \) for some \( j \) by Proposition 2.1, hence \( J = P_j \). So \( \text{rk} J = 0 \) which is a contradiction. So \( R \) is an integral domain.

Now we look at some consequences of this theorem.

**Exercise 12.2.J** Suppose \( p \) is a regular point of a Noetherian scheme \( X \). Show that only one irreducible component of \( X \) passes through \( p \).

**Proof.** Since the minimal primes of the local ring \( \mathcal{O}_{X,p} \) correspond to the irreducible components passing through \( p \). As \( \mathcal{O}_{X,p} \) is an integral domain by Theorem 2.1, there is only one irreducible component passing through \( p \). \( \square \)
Easy Exercise 12.2.K Show that a nonempty regular Noetherian scheme is irreducible if and only if it is connected.

Proof. This follows from previous exercise and note that every connected component of a topological space $X$ is the union of irreducible components.

Important Exercise 12.2.K (Regular Schemes in Regular Schemes are Regular Embeddings).

(a) Suppose $(A, \mathfrak{m}, k)$ is a regular local ring of dimension $n$, and $I \subset A$ is an ideal of $A$ cutting out a regular local ring of dimension $d$. Let $r = n - d$. Show that $\text{Spec } A/I$ is a regular embedding in $\text{Spec } A$. Hint: show that there are elements $f_1, \ldots, f_r$ of $I$ spanning the $k$-vector space $I/(I \cap \mathfrak{m}^2)$. Show that the quotient of $A$ by both $(f_1, \ldots, f_r)$ and $I$ yields dimension $d$ regular local rings. Show that a surjection of integral domains of the same dimension must be an isomorphism.

(b) Suppose $\pi : X \to Y$ is a closed embedding of regular schemes. Show that $\pi$ is a regular embedding.

Proof. Since $A/I$ is regular local with maximal ideal $\mathfrak{m}/I$, we have that $\dim(\mathfrak{m}/I)/(\mathfrak{m}/I)^2 = r$. Since $(\mathfrak{m}/I)/(\mathfrak{m}/I)^2 \cong I/(I \cap \mathfrak{m}^2)$, there are $f_1, \ldots, f_r \in I$ as a basis for the $k$-vector space $I/(I \cap \mathfrak{m}^2)$. Note that $R/(f_1, \ldots, f_r)$ is a regular local ring of dimension $d$ and $f_1, \ldots, f_r$ is a regular sequence. To see this, note that since $f_1$ is not a zerodivisor of $R$ ($R$ is an integral domain), $R/(f_1, \ldots, f_r)$ is regular local of dimension $n - 1$ and proceed with the same argument. Then we have a surjection of regular local rings of the same dimension $R/(f_1, \ldots, f_r) \to R/I$ which must be an isomorphism. (If not, then the kernel is nontrivial, then we would have $\dim R/(f_1, \ldots, f_r) \geq \dim R/I + 1$ since the trivial contain the zero ideal which is prime in a domain.) Thus we have shown (a) that $\text{Spec } A/I$ is regular embedding. (b) follows from (a) as we have shown it locally.

References
