Applications of Krull's Principal Ideal Theorem

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This post is about some applications of Krull's Principal Ideal Theorem and regular local rings in dimension theory and regularity of schemes [Part IV, Vakil], with the aim of connecting the 2018-2019 Warwick course MA4H8 Ring Theory with algebraic geometry. The lecture notes/algebraic references are here: 2018-2019 Ring Theory. Note that the algebraic results included here follow the notes. Alternatively, one can also find them in [Vakil, [1]] either as exercises or proved results for which I have included the references.

Besides including results in both their geometric and algebraic statements, I have given proofs to a selection of exercises in Part IV, [Vakil, [1]] to illustrate more applications and other connections to the contents in the Ring Theory courses. The indexes for exercises follow those in [Vakil, [1]].

1 Krull's Principal Ideal Theorem

Krull's Principal Ideal Theorem (algebraic version).[P.23, [2]] Let R be a Noetherian ring. Let $a \in R$ be a non unit, suppose that P is a prime ideal minimal over a. Then $\operatorname{rk} P \leq 1$. If furthermore, a is not a zero divisor, then $\operatorname{rk} P = 1$.
Krull's Principal Ideal Theorem (geometric version).[P. 316, [1]] Suppose X is a locally Noetherian scheme, and f is a function. The irreducible components of $V(f)$ are codimension 0 or 1.

See [P.23, [2]] for the proofs.
The Generalised Principal Ideal Theorem. Let R be a commutative Noetherian Ring. Suppose that P is a prime ideal minimal over the elements $x_1, \dots, x_r \in R$, then $\operatorname{rk} P \leq r$.

Proof. We prove by induction. For $r = 1$, this is Krull's Principal Ideal Theorem.

Assume the result is true for primes minimal over $\leq r-1$ elements. Suppose P is minimal over $x_1, \dots x_r$ and suppose there exists a chain of primes $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{r+1}$. If $x_1 \in P_R$ then since P_0/x_1R is minimal over $\bar{x_2}, \dots, x_r$ in the ring R/x_1R we get a contradiction.

Let k be such that $x_1 \in P_k$ but $x_1 \notin P_{k+1}$. So we have $P_k \supseteq P_{k+2} + x_1R \supsetneq P_{k+2}$ hence a chain $P_k/P_{k+2} \supseteq x_1 + P_{k+2} \supsetneq P_{k+2}/P_{k+2}$. Since $P_k/P_{k+2} \supsetneq P_{k+1}/P_{k+2} \supsetneq P_{k+2}/P_{k+2}$, by Krull's Principal Ideal Theorem, P_k/P_{k+2} can not be minimal over $x_1 + P_{k+2}$. So there exists a prime ideal P'_{k+1} such that $P_k \supsetneq P'_{k+1} \supseteq x_1 + P_{k+2} \supsetneq P_{k+2}$. Proceeding this way, we can build a new chain $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_k \supsetneq P'_{k+1} \supsetneq \dots P'_r \supsetneq P_{r+1}$ with $x_1 \in P'_r$ which leads to a contradiction as before. □

Exercise 11.3.A Show that an irreducible homogeneous polynomial in $n+1$ variables over a field k describes an integral scheme of dimension $n-1$.

Proof. Let $f(x_0, \dots, x_n)$ be the homogeneous equation cut out the scheme in question. Note that it is covered by $\operatorname{spec} k[x_{0/i}, \dots, \widehat{x_{i/i}}, \dots x_{n/i}]/(f(x_{0/i}, \dots, \widehat{x_{i/i}}, \dots x_{n/i}))$ and f is irreducible, then apply Krull's Principal Ideal Theorem. □

Exercise 11.3.B. *[Lemma 4.38,[2]] Suppose* $(R, \mathfrak{m})$ *is a Noetherian local ring, and* $f \in \mathfrak{m}$*. Then* $\dim R/(f) \geq \dim R - 1$*.*

Proof. Let $n = \dim A$, then $\operatorname{rk}\mathfrak{m} = n$, so there exists a chain of primes $\mathfrak{m} = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$. As in the proof of the Generalised Principal Ideal Theorem, we can construct a new chain of primes $\mathfrak{m} = Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_{n-1}$ with $f \in Q_{n-1}$. Hence, $\operatorname{rk}(\mathfrak{m}/(f)) \geq n-1$.

We can further more show that if f is regular(nonzerodivisor) then the equality holds. If $\mathfrak{m}/cR = T_0/(f) \supsetneq \cdots \supsetneq T_k/(f)$ is a chain of primes in $R/(f)$ then $J = T_0 \supsetneq T_1 \supsetneq \cdots \supsetneq T_k$ is a chain of primes in R. Since f is regular, T_k is not a minimal prime of R as $f \in R$ (The regular elements of R is the complement of the union of all minimal primes, see Proposition 4.18, [2]). So $n = \operatorname{rk}\mathfrak{m} = \operatorname{rk}\mathfrak{m}/(f) + 1$. Hence, $\operatorname{rk}\mathfrak{m}/(f) = n-1$. □

Important Exercise 11.3.C (a) (Hypersurfaces meet everything of dimension at least 1 in projective space, unlike in affine space.) Suppose X is a closed subset of $\mathbb{P}_n^k$ of dimension at least 1, and H is a nonempty hypersurface in $\mathbb{P}_n^k$. Show that H meets X.

(b) Suppose $X \hookrightarrow \mathbb{P}_n^k$ is a closed subset of dimension r. Show that any codimension r linear space meets X.

(c) Show further that there is an intersection of $r+1$ nonempty hypersurfaces missing X. If k is infinite, show that there is a codimension $r+1$ linear subspace missing X.

(d) If k is an infinite field, show that there is an intersection of r hyperplanes meeting X in a finite number of points.

Proof. (a) Consider the affine cone $CX \subseteq \mathbb{A}_k^{n+1}$ over X. Let $\dim X = n$, then $\dim CX = n+1$, by Krull's Principal Ideal Theorem, $\dim CX \cap V(f) \geq n+1-1 = n \geq 1$. Note that $\dim CX \cap V(f)$ contains the origin. Since it has dimension ≥ 1, it also contains another point in $\mathbb{A}_k^{n+1} \setminus \{0\} \hookrightarrow \mathbb{P}_k^n$.

(b) Similar to (a), note that $\dim CX \cap V(f_1, \ldots, f_n) \geq \dim X + 1 - r \geq 1$.

(c) The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain any generic point of X. Show this by induction on the number of generic points. To get from m to $m+1$: take a hypersurface not vanishing on $p_1, \ldots, p_m$. If it doesn't vanish on p_{m+1}, we are done. Otherwise, call this hypersurface f_{m+1}. Do something similar with $m+1$ replaced by i for each $1 \leq i \leq m$. Then consider $\Sigma_i f_1 \cdots \hat{f_i} \cdots f_{m+1}$, this hypersurface doesn't vanish at any p_m.

Let $X_{-1} = X$ and $g_{-1} = 0$, take g_i to be a hypersurface that doesn't contain any generic point of $X_i = X_{i-1} \cap V(g_{i-1})$. Then by Krull's principal Ideal Theorem, $\dim X_i = \dim X_{i-1} - 1$. So we have $\dim X_{r+1} = \dim X - (r+1) = -1$, thus the intersection of the $r+1$ hypersurfaces g_i and X is empty.

If k is finite, we can choose a closed point $[a_0 :, \ldots a_n]$ for each irreducible component and choose a homogeneous linear form that does not vanish at any of these points.

(d) As in (c), we can take r linear forms g_i such that $\dim X_r = 0$ which means the intersection of X with the r hyperplanes consists of finitely many points. Note that a Noetherian ring with dimension 0 has only finitely many primes. □

2 Regularity and Smoothness

2.1 Regular Local Ring

[4.7, [2]] Let R be a Noetherian local ring with Jacobson radical (the maximal ideal) J. We have $V(R) := \dim J/J^2$ as a vector space over the field R/J. So $V(R) =$ the number of elements in a minimal generator set for J. By the Generalised Principal Ideal Theorem, we have $\operatorname{rk} J \leq V(R)$ [also see Theorem 12.2.1, [1]].

(Fact(Cor 2.23,[2]): $x_1, \dots x_k$ is a minimal generating set for $J \iff \bar{x_1} \dots \bar{x_k}$ is a basis for the vector space J/J^2 over R/J.)

Definition 2.1. *A Noetherian local ring is called a* ***regular local ring*** *if* $\operatorname{rk} J = V(R)$.

Important Exercise 12.1.B. *[Krull's Principal Ideal Theorem for Tangent Spaces]. Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the Zariski tangent space of $A/(f)$ is cut out in the Zariski tangent space of A by $f(\mod \mathfrak{m}^2)$. Hence the dimension of the Zariski tangent space of* $\operatorname{Spec} A/(f)$ *at* $[\mathfrak{m}]$ *is the dimension of the Zariski tangent space of* $\operatorname{Spec} A$ *at* $[\mathfrak{m}]$*, or one less.*

Note that localization and quotient commutes, we can translate the above into the following algebraic statement.

Algebraic translation[Lemma 4.40, [2]]. Let R be a Noetherian local ring with Jacobson radical J (R not a field). Suppose that $x \in J \setminus J^2$, let $R^* = R/xR$. Then $V(R^*) = V(R) - 1$. (If $x \in J^2$, then $V(R^*) = V(R)$.)

Proof. Note that R^* is a Noetherian local ring with Jacobson radical $J^* = J/xR$. Let $y_1^*, \dots y_k^*$ be a minimal generating set for J^*. Choose $y_1, \dots y_k \in J$ such that $y_i \mapsto y_i^*$ under the natural homomorphism $R \to R/xR$. Claim $x, y_1, \dots y_k$ is a minimal generating set for J which is equivalent to that $\bar{x}, \bar{y_1}, \dots \bar{y_k}$ is a basis for the vector space J/J^2 over R/J. We shall now show that $\bar{x}, \bar{y_1}, \dots \bar{y_k}$ in the vector space J/J^2 are linearly independent.

Suppose that $xr + y_1 r_1 + \dots + y_k r_k \in J^2$. So $y_1^* r_1^* + \dots + y_k^* r_k^* \in (J^*)^2$ where r_i^* are the homomorphic images of r_i under $R \to R/xR$. Since $y_1^*, \dots y_k^*$ is a minimal generating set for J^*, $\bar{y_1^*}, \dots \bar{y_k^*}$ is a basis for the vector space $J^*/(J^*)^2$ over R/J^*. It follows that $r_i^* \in J^*$, so $r_i \in J$. Then $xr \in J^2$. If $r \notin J$ then r is a unit, it follows that $x \in J^2$, a contradiction. Thus $r \in J$. This proves the claim. (If $x \in J^2$, from the above we can easily see that the claim becomes that $\bar{y_1}, \dots \bar{y_k}$ is a basis for the vector space J/J^2.) □

We now look at some applications of this results:

Exercise 12.1.D Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a codimension 1 ideal that is not principal, using the method of Solution 12.1.4. (See Figure 12.2 for the projectivization of this situation — a line on a smooth quadric surface.)

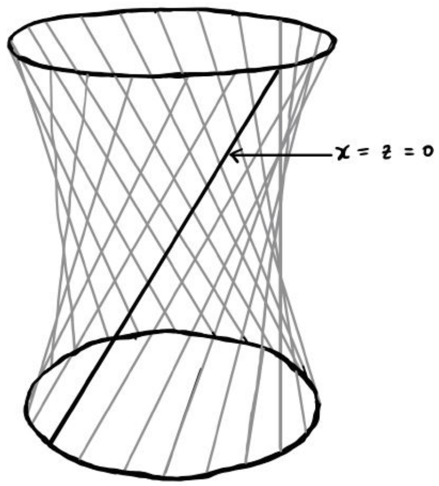

FIGURE 12.2. The line $V(x, z)$ on the smooth quadric surface $V(wz - xy) \subset \mathbb{P}^3$.

Proof. Let $A = k[w,x,y,z]/(wz - xy)$, then $\dim A = 3$ by Krull's Principal Ideal Theorem. As $A/(x,y) \cong k[w,y]$, $\dim A/(x,y) = 2$, so (x,z) has codimension 1. And $\operatorname{Spec} A/(x,y)$ has Zariski tangent space of dimension 2 at the origin. But $\operatorname{Spec} A/(f)$ must have Zariski tangent space of dimension at least 3 by Exercise 12.1.B. □

Exercise 12.1.E Let $A = k[w,x,y,z]/(wz - xy)$. Show that $\operatorname{Spec} A$ is not factorial.

Proof. This follows from Exercise 12.1.D and that all codimension 1 prime ideal in a unique factorisation domain is principal (Lemma 11.1.6, [1]) . □

Remark. *A related algebraic result is Proposition 7.5, [2]: Let R be a commutative Noetherian integral domain. Then R is a UFD if and only if every rank 1 prime ideal of R is principal.*

Remark (Vakil)**.** *As A is integrally closed if $\operatorname{char} k \neq 2$ (Exercise 5.4.I(c),[1]), this yields an example of a scheme that is normal but not factorial.*

Exercise 12.2.B Suppose X is a finite type k-scheme (such as a variety) of pure dimension n, and p is a nonsingular closed point of X, so $\mathcal{O}_{X,p}$ is a regular local ring of dimension n. Suppose $f \in \mathcal{O}_{X,p}$. Show that $\mathcal{O}_{X,p}/(f)$ is a regular local ring of dimension $n-1$ if and only if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Proof. In the algebraic translation of Important Exercise 12.1.B, we have that $V(R^*) = V(R) - 1$ if $x \in J \setminus J^2$; and $V(R^*) = V(R)$ if $x \in J^2$. Now we continue this algebraic statement, we show that for $x \in J/J^2$,

$$V(R) - 1 = V(R^*) \geq \operatorname{rk} J^* \geq \operatorname{rk} J - 1 (\text{by Exericise 11.3.B}) = V(R) - 1.$$

So $V(R^*) = \operatorname{rk} J^*$ and R^* is regular local.

If $x \in J^2$, in the case of the exercise 12.2.B (X is a finite type k-scheme), x is not a zero divisor, so we still have $\operatorname{rk} J^* = \operatorname{rk} J - 1$. Since $V(R) = V(R^*) = \operatorname{rk} J$, we see that in this case R^* is not regular local. This proves the exercise. □

Recall the definition of Cartier divisor [8.4.1,[1]]: If $X \to Y$ is a closed embedding, and there is a cover of Y by affine open subsets $\operatorname{Spec} A_i \subset Y$, and there exist non-zerodivisors $t_i \in A_i$ with $V(t_i) = X_i|_{\operatorname{Spec} A_i}$(scheme-theoretically—i.e., the ideal sheaf of X over $\operatorname{Spec} A_i$ is generated by t_i), then we say that X is an **effective Cartier divisor** on Y.

Exercise 12.2.C [The Slicing Criterion for Regularity] Suppose X is a finite type k-scheme (such as a variety), D is an effective Cartier divisor on X (Definition 8.4.1), and $p \in D$. Show that if p is a regular point of D then p is a regular point of X.

Proof. The algebraic translation is that if $R/(f)$ is regular local for f regular (a nonzerodivisor), then R is also regular local. We proceed with a similar argument as the last exercise.

$$\begin{aligned} V(R) - 1 &\leq V(R^*) \quad (\text{by Important Ex 12.1.B}) \\ &= \operatorname{rk} J^* = \operatorname{rk} J - 1 \leq V(R) - 1. \end{aligned}$$

So $\operatorname{rk} J = V(R)$ and R is regular. □

2.2 Regular local rings are integral domains

Proposition 2.1 (Lemma 4.19, [2]). *Let R be a commutative ring. Let $P_1, \dots, P_n$ be ideals of R, at least $n-2$ of which are prime. Let S be a subring of R. Suppose that $S \subseteq \bigcap_{i=1}^n P_i$, then $S \subseteq P_k$ for some k, $1 \le k \le n$. (Note that S does not necessarily contain 1 since we do not assume R is unital.)*

Proof. Prove by induction on n. For $n = 1$, the result is trivial. For $n = 2$ if $S \nsubseteq P_1$ and $S \nsubseteq P_2$ then choose $x_1, x_2 \in S$ such that $x_1 \notin P_2$ and $x_2 \notin P_1$. Then $x_1 + x_2 \in S$ but $x_1 + x_2 \notin P_i$, $i = 1, 2$.

Now assume $n > 2$ and that the result holds for values $< n$. Clearly any selection of $n-1$ of the P_i at most 2 will be non-prime. Suppose that $S \subseteq \bigcap_{i=1}^n P_i$ but $S \nsubseteq P_i$ for any $1 \le i \le n$. Then $S \nsubseteq \bigcap_{i=1, i\neq k}^n P_i$. Thus $x_k \in P_k$. Since $n > 2$, at least one of the P_i must be prime, say P_1. Let $y = x_1 + x_2 \cdots x_n$, then $y \notin P_i$ for any $1 \le i \le n$, a contradiction. This completes the induction. □

Corollary 2.1 (Prime Avoidance, Prop 11.2.13,[1]). *Suppose $p_1, \dots, p_n$ are prime ideals of a ring A, and I is another ideal of A not contained in any p_i. Then I is not contained in $\cap p_i$: there is an element $f \in I$ not in any of the p_i.* □

Lemma 2.1 (Lemma 4.42,[2]). *Let R be a Noetherian local ring which is not an integral domain. Let $P = pR(p \in P)$ be a prime ideal. Then $\operatorname{rk} P = 0$.*

Proof. Suppose that $Q \subsetneq P$ where Q is a prime ideal, then $p \notin Q$. Now $q \in Q$ implies $q = pt$ for some $t \in R$. Hence $pt \in Q \Rightarrow t \in Q$ since $p \notin Q$. So $q \in pQ \subseteq P^2 \subseteq p^2 R$. Preceding this way we have $Q \subseteq P^n$ for all $n \ge 1$, so $Q \subsetneq \bigcap_{n=1}^\infty P^n \subseteq \bigcap_{n=1}^\infty J^n$ where J is the Jacobson radical of R. But by Theorem 4.9, [2] $\bigcap_{n=1}^\infty J^n = 0$, so $Q = 0$ which is a contradiction since R is not a domain. Hence $\operatorname{rk} P = 0$. □

Theorem 2.1 (Theorem 4.43,[2]/Theorem 12.1.13,[1]). *A regular local ring is an integral domain.*

Proof. By induction on $\dim R = \operatorname{rk} J$. If $\operatorname{rk} J = 0$ then R must be a field.

Suppose now that $\operatorname{rk} J = n > 0$ and assume result for rings of dim $< n$. Since $V(R) = \dim J/J^2 = \operatorname{rk} J \neq 0$, $J \neq J^2$. Choose $x \in J \setminus J^2$. By Exercise 12.2.B or Theorem 4.41,[2], $R^* = R/xR$ is regular local. Also $\dim R^* = \dim R - 1$. By induction hypothesis, R^* is an integral domain, that is, xR is a prime ideal. Suppose that R is not an integral domain, then by Lemma 2.1, xR is a minimal prime. Let $P_1, \dots P_k$ be the minimal primes of R, then $x \in P_1 \cap \dots P_k$, thus $J \setminus J^2 \subseteq P_1 \cap \dots P_k$, $J \subseteq J^2 \cap P_1 \cap \dots \cap P_k$. So $J \subseteq P_j$ for some j by Proposition 2.1, hence $J = P_j$. So $\operatorname{rk} J = 0$ which is a contradiction. So R is an integral domain. □

Now we look at some consequences of this theorem.

Exercise 12.2.J Suppose p is a regular point of a Noetherian scheme X. Show that only one irreducible component of X passes through p.

Proof. Since the minimal primes of the local ring $\mathcal{O}_{X,p}$ correspond to the irreducible components passing through p. As $\mathcal{O}_{X,p}$ is an integral domain by Theorem 2.1, there is only one irreducible component passing through p. □

Easy Exercise 12.2.K Show that a nonempty regular Noetherian scheme is irreducible if and only if it is connected.

Proof. This follows from previous exercise and note that every connected component of a topological space X is the union of irreducible components. □

Important Exercise 12.2.K (Regular Schemes in Regular Schemes are Regular Embeddings).

(a) Suppose $(A, \mathfrak{m}, k)$ is a regular local ring of dimension n, and $I \subset A$ is an ideal of A cutting out a regular local ring of dimension d. Let $r = n - d$. Show that $\operatorname{Spec} A/I$ is a regular embedding in $\operatorname{Spec} A$. Hint: show that there are elements $f_1, \dots, f_r$ of I spanning the k-vector space $I/(I \cap \mathfrak{m}^2)$. Show that the quotient of A by both $(f_1, \dots, f_r)$ and I yields dimension d regular local rings. Show that a surjection of integral domains of the same dimension must be an isomorphism.

(b) Suppose $\pi : X \to Y$ is a closed embedding of regular schemes. Show that π is a regular embedding.

Proof. Since A/I is regular local with maximal ideal $\mathfrak{m}/I$, we have that $\dim(\mathfrak{m}/I)/(\mathfrak{m}/I)^2 = r$. Since $(\mathfrak{m}/I)/(\mathfrak{m}/I)^2 \cong I/(I \cap \mathfrak{m}^2)$, there are $f_1, \dots f_r \in I$ as a basis for the k-vector space $I/(I \cap \mathfrak{m}^2)$. Note that $R/(f_1, \dots, f_r)$ is a regular local ring of dimension d and $f_1, \dots, f_r$ is a regular sequence. To see this, note that since f_1 is not a zerodivisor of R (R is an integral domain), $R/(f_1)$ is regular local of dimension $n - 1$ and proceed with the same argument. Then we have a surjection of regular local rings of the same dimension $R/(f_1, \dots f_r) \to R/I$ which must be an isomorphism. (If not, then the kernel is nontrivial, then we would have $\dim R/(f_1, \dots f_r) \geq \dim R/I + 1$ since the trivial contain the zero ideal which is prime in a domain.) Thus we have shown (a) that $\operatorname{Spec} A/I$ is regular embedding. (b) follows from (a) as we have shown it locally. □

References

[1] Vakil, R. (2017). *The Rising Sea: Foundations of Algebraic Geometry.* Preprint.

[2] Bouyer, F (2011). *Ring Theory (MA4H8)*, lecture notes, University of Warwick, delivered by C. Hajarnavis Oct-Dec 2018: Online Notes.