Lusin's Theorem and Continuous Extension

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Here we give proofs for two versions of Lusin's Theorem, one from Exercise 44, Ch 2 in [1] and the other from [2] the textbook used for my undergraduate mathematical analysis course in Beijing. The latter version is a stronger result which in addition discusses the condition for a real-valued function defined on a subset of $\mathbb{R}^n$ to be continuously extended to the whole of $\mathbb{R}^n$. A more general result in topology is the Tietze Extension Theorem.

1 Lusin's Theorem

Theorem 1.1 (**Egoroff's Theorem**, 2.33, [1]). *Suppose that μ is a finite measure and $f_k \to f$ a.e. Then for every $\epsilon > 0$, there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.*

Proof. Let

$$E_{nk} = \bigcup_{m=n}^{\infty} \{x : |f_m(x) - f(x)| \geq 1/k\}.$$

For fixed k, E_{nk} decreases as n increases. The intersection $\bigcap_n E_{nk}$ has measure 0 because for almost every x, $|f_m(x) - f(x)| < 1/k$ if m is sufficiently large. Therefore, $\mu(E_{nk}) \to 0$ as $n \to \infty$. We can thus find an integer n_k such that $\mu(E_{n_k k}) < \epsilon 2^{-k}$. Let

$$E = \bigcup_{k=1}^{\infty} E_{n_k k}.$$

Hence $\mu(E) < \epsilon$. If $x \notin E$, then $x \notin E_{n_k k}$ for some k, so $|f_n(x) - f(x)| < 1/k$ if $n \geq n_k$. Thus $f_n \to f$ uniformly on E^c. □

In the rest of the post, we denote by μ the Lebesgue measure on $\mathbb{R}^n$.

Theorem 1.2 (**Lusin's Theorem**(Version 1), Ex 44, Ch2, [1]). *Suppose $E \subset \mathbb{R}^n$ is Lebesgue measurable, $f : E \to \mathbb{R}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $F \subset E$ such that $\mu(F^c) < \epsilon$ and $f|_F$ is continuous.*

Proof. Recall that Theorem 1.18 [P.36, [1]] says that if E is μ-measurable, then

$$\mu(E) = \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open}\} = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

This means we can approximate a measurable set by open or compact sets. For a simple function $\phi = \Sigma_{k=1}^m a_k \chi_{E_k}$, and that the E_k are disjoint and measurable. Fix $\epsilon > 0$, there exists a compact set $F_k \subset E_k$ such that $\mu(E_k \setminus F_k) < \epsilon/m$. Let $F = \bigcup_{k=1}^m F_k$, then since f_{F_k} is continuous with F_k closed and disjoint, $f|_F$ is continuous.

Now let $f : E \to \mathbb{R}$ be arbitrary and ϕ_k be simple functions such that $\phi_k(x) \to f(x)$ for any $x \in E$. Fix $\epsilon > 0$, by Egoroff's Theorem 1.1, $\forall k \in \mathbb{N}^+$, $\exists$ compact $F_0 \subset E$ such that

(i) $\mu(E \setminus F_0) < \epsilon/2$;

(ii) $f_k \to f$ uniformly on F_0.

By the above discussion for simple functions, $\forall k \in \mathbb{N}^+$, $\exists$ compact $F_k \subset E$ such that:

(iii) $\mu(E \setminus F_k) < \epsilon/2^{k+1}$;

(iv) f_k restricted to F_k is continuous.

Let $F = \bigcap_{k=0}^{\infty} F_k$, then F is compact and $\mu(E \setminus F) < \epsilon$, $f_k \to f$ uniformly on F. Since f_k is continuous on F for any k, we have that f is continuous on F[1]. □

2 Continuous Extension

2.1 Lusin's Theorem with Continuous Extension

Let $C(E)$ denote the space of continuous function on E.

Theorem 2.1 (Lusin's Theorem(Version 2), Theorem 10.2.6, P331, [2])**.** *Suppose $E \subset \mathbb{R}^n$ is Lebesgue measurable and $f : E \to \bar{\mathbb{R}}$ is a Lebesgue measurable extended real valued function with $\mu(|f| = +\infty) = 0$, then $\forall \epsilon > 0$, $\exists g \in C(E)$ such that $\mu(f \neq g) < \epsilon$.*

Proof. Along with the proof of Theorem 1.2, one in addition needs Theorem 2.2 to finish the proof. □

2.2 Continuous Extension

Lemma 2.1 (Lemma 10.2.1, P331[2])**.** *Suppose $E \subset \mathbb{R}^n$ is compact, $f \in C(E)$, then there exists an increasing continuous function on $[0, +\infty]$ such that:*

(1) $\bar{\omega}(0) = 0$;

(2) $\forall r, x \geq 0$, $\bar{\omega}(r + s) \leq \bar{\omega}(r) + \bar{\omega}(s)$;

(3) $\forall x, y \in E$, $|f(x) - f(y)| \leq \bar{\omega}(|x - y|)$.

Proof. Since E is compact, $f \in C(E)$, define

$$\omega(r) = \max\{|f(x) - f(y)| : |x - y| \leq r, x, y \in E, \forall r \geq 0\}.$$

One can check that

(i) $\lim_{r\to 0^+} \omega(r) = 0$;

(ii) ω is increasing on $[0, +\infty]$;

(iii) $\forall x, y \in E$, $|f(x) - f(y)| \leq \omega(|x - y|)$;

(iv) $\forall r \geq \operatorname{diam} E = \max\{|x-y| : x, y \in E\}$, $\omega(r) = C(f) = \max\{f(x)-f(y) : x, y \in E\}$.

Let $\mathcal{A} = \{L : L \text{ is a linear function on } \mathbb{R} \text{ such that } \omega(r) \leq L(r), \forall r \geq 0\}$. By (iv), $\mathcal{A} \neq \emptyset$. Let

$$\bar{\omega}(r) = \inf\{L(r) : L \in \mathcal{A}\}, \forall r \geq 0.$$

Then $\bar{\omega}$ is the desired function. □

Theorem 2.2 (Theorem 10.2.7, P332, [2])**.** *Suppose $E \subset \mathbb{R}^n$, then f can be extended to a continuous function on $\mathbb{R}^n$ if and only if f can be extended to a continuous function on the closure $\bar{E}$ of E.*

[1]To see this, for any $\epsilon > 0$, $\exists N \in \mathbb{N}^+$, such that $\forall k \geq N$, $|f_k(x) - f(x)| < \epsilon/3$, $\forall x \in F$. Fix $\hat{x} \in F$, the continuity of f_N gives $\delta > 0$ such that $|F_N(x) - f_N(\hat{x})| < \epsilon/3$, $\forall x \in U(\hat{x}, \delta) \cap F$, thus $\forall x \in U(\hat{x}, \delta) \cap F$, $|f(x) - f(\hat{x})| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(\hat{x})| + |f_N(\hat{x}) - f(x)| < \epsilon$, thus f is continuous.

Proof. The only if direction is obvious. We show the if direction. We may assume that $E = \bar{E}$ and f is continuous on E.

Step1. When E is bounded, that is, E is compact, then there is an increasing continuous function $\bar{\omega}$ satisfying (1)-(3) in Lemma 2.1.

Claim. Define

$$g(x) = \min\{f(y) + \bar{\omega}(|x-y|) : y \in E\}, \forall x \in \mathbb{R}^n,$$

then g satisfies:

(1) $|g(x) - g(z)| \leq \bar{\omega}(|x-y|)$, $\forall x, y \in \mathbb{R}^n$; (2) $g(x) = f(x)$, $\forall x \in E$.

Proof of Claim. Take $x, z \in \mathbb{R}^n$, we have

$$\begin{aligned} g(x) - g(z) &= \min\{f(y) + \bar{\omega}(|x-y|) : y \in E\} - \min\{f(y) + \bar{\omega}(|z-y|) : y \in E\} \\ &\leq \max\{\bar{\omega}(|x-y|) - \bar{\omega}(|z-y|) : y \in E\} \\ &\leq \max\{\bar{\omega}(|x-y| - |z-y|) : y \in E\} \leq \bar{\omega}(|x-z|). \end{aligned}$$

Similarly, $g(z) - g(x) \leq \bar{\omega}(|z-x|) = \bar{\omega}(|x-z|)$. Thus $|g(x) - g(z)| \leq \bar{\omega}(|x-z|)$.

For any $x \in E$, $g(x) \leq f(x) + \bar{\omega}(|x-x|) = f(x)$. On the other hand, there exists $\hat{y} \in E$, $g(x) = f(\hat{y}) + \bar{\omega}(|x-\hat{y}|) \geq f(\hat{y}) + f(x) - f(\hat{y}) = f(x)$ where we use the inequality $|f(x) - f(y)| \leq \omega(|x-y|)$. □

Thus we have shown that g is a continuous extension of f on $\mathbb{R}^n$.

Step2. For the general case, let $E_1 = E \cap \bar{U}(0,1)$ and $E_k = E \cap \bar{U}(0,k) \setminus \bar{U}(0,k-1)$, $k \in \mathbb{N}^+$, where $\bar{U}(0,k) = \{x \in \mathbb{R}^n : |x| \leq k\}$. We may assume $E_1 \neq \emptyset$. Let f_1 be the restriction of f on E_1, then from step 1, there exists $g_1 \in C(\mathbb{R}^n)$ with $g_1 = f_1$ on E_1.

Then define f_2 on $F_2 = \bar{U}(0,1) \cup E_2$:

$$f_2(x) = \begin{cases} f(x) \text{ if } x \in E_2 \\ g_1(x) \text{ if } x \in \bar{U}(0,1). \end{cases}$$

It can be seen that (i) $f_2 \in C(F_2)$, (ii) F_2 is compact. Then we can repeat the same procedure for f_2 to get $g_2 \in C(\mathbb{R}^n)$ and so on. So by induction, we have a sequence of continuous function g_k on $\mathbb{R}^n$ with the following properties:

(1) $g_k(x) = f(x)$, $\forall x \in E_k$, $k \in \mathbb{N}^+$; (2) $g_{k+1}(x) = g_k(x)$, $\forall x \in \bar{U}(0,k)$, $k \in \mathbb{N}^+$.

Define $g : \mathbb{R}^n \to \mathbb{R}$ as

$$g(x) = \lim_{k \to +\infty} g_k(x), \forall x \in \mathbb{R}^n.$$

Note that $\forall x \in \mathbb{R}^n$, when $k > [|x|]$, $g_k(x) = g(x)$, so $g \in C(\mathbb{R}^n)$ and $\forall x \in E$, $g(x) = f(x)$. □

Remark. *More generally, for real valued functions on a topological space, there is the* ***Tietze Extension Theorem****: Let X be normal and $F \subset X$ be closed and let $f : F \to R$ be continuous. Then there is a map $g : X \to R$ such that $g(x) = f(x)$ for all $x \in F$. (Note that in topology, by a map we mean a continuous function.)*

References

[1] Folland, G. B. (1999). *Real analysis: modern techniques and their applications* (Vol. 40). John Wiley & Sons.

[2] Kunyang Wang, Zhongdan Huan& Yongping Liu. (2009). *Concise Mathematical Analysis*(in Chinese). Higher Education Press.