each $i \in J$ and the value 1_C at each arrow of J. A natural transformation $\tau : F \to \Delta c$ between a functor $F \in C^J$ and a $c \in C$ can be represented as a commutative diagram, called a **cone** from the base F to the vertex c [[3], p. 67]

$$\begin{array}{ccccc} F_i & \xrightarrow{F_u} & F_j & \xrightarrow{F_i} & F_k \\ & \searrow^{\tau_i} & \downarrow{\scriptstyle \tau_j} & \swarrow_{\tau_k} & \\ & & c & & \end{array} \quad , \tag{A.1}$$

where F_i is just the value of F at i.

Definition A.2.1. *A **colimit** of $F : J \to C$ consists of an object $\underset{\rightarrow}{Lim}F \in C$ and a cone $\mu : F \to \Delta(\underset{\rightarrow}{Lim}F)$ which is universal: For any cone $\tau : F \to \Delta c$ from the base F, there is a unique arrow $t' : \underset{\rightarrow}{Lim}F \to c$ with $\tau_i = t'\mu_i$ for every index $i \in J$.*

Below are theorems about constructing colimits from coproducts and coequalisers which are the dual versions of the original theorems [[3], Theorem 1 and 2 on p. 113]. The set of all objects and the set of all arrows are denoted as $ob(J)$ and $ar(J)$.

Theorem A.2.1. *For categories C and J, if C has coequalisers of all pairs of arrows and all coproducts indexed by the sets $ob(J)$ and $ar(J)$, then C has a colimit for every functor $F : J \to C$.*

Proof Let $\coprod_i F_i$ be the coproduct taken over all objects in J, and $\coprod_u F_{dom\,u}$ be the coproduct taken over all arrows u of J with argument at each arrow u the value $F_k = F_{dom\,u}$ of F at the codomain of u. Then by assumption, these two coproducts exist. Since $\coprod_u F_{dom\,u}$ is a coproduct, there is a unique arrow f such that the upper square commutes for every u and a unique arrow g such that the lower square commutes for every u in the diagram

$$\begin{array}{ccccc} F_{dom\,u} & = = = & F_{dom\,u} & & F_i \\ \downarrow{\scriptstyle i_u} & & \downarrow{\scriptstyle i_{dom\,u}} & \swarrow^{i_i} & \downarrow{\scriptstyle \mu_i} \\ \coprod_u F_{dom\,u} & \overset{f}{\underset{g}{\rightrightarrows}} & \coprod_i F_i & \overset{e}{\dashrightarrow} & d \\ \uparrow{\scriptstyle i_u} & & \uparrow{\scriptstyle i_{cod\,u}} & & \\ F_{dom\,u} & \xrightarrow[F_u]{} & F_{cod\,u} & & \end{array} \quad . \tag{A.2}$$

By hypothesis, there exists an coequaliser e for f and g. Its composite with the injections i_i give arrows $\mu_i = ei_i : F_i \to d$ for each i. Since e coequalises f and g and the two square in the diagram commutes, one has $\mu_j F_u = \mu_k$. for every $u : j \to k$, hence $\mu : \Delta d \to F$ is a cone from the vertex d to the base F. If τ is any other such cone of vertex c whose maps τ_i combine to yield a unique map $h : \coprod_i F_i \to c$, then $hf = hg$. Hence h factors uniquely through u and therefore the cone τ factors uniquely through the cone μ. This provides that d and the cone μ provide a colimit for F. $\square$

Theorem A.2.2 (Colimits by coproducts and coequalisers)**.** *The colimit of* $F : J \to C$ *is the coequaliser* e *of* $f, g : \prod_u F_{dom\,u} \to \prod_i F_i$ *(*$u \in ar(J)$*,* $i \in J$*), where* $fi_u = p_{dom\,u}$*,* $gi_u = i_{cod\,u} F_u$*, the limiting cone* μ *is* $\mu_j = ei_j$ *for* $j \in J$*, all as in (A.2).*

Below is a useful fact about colimits in functor categories:

Proposition A.2.1 ([1], p. 41)**.** *In a functor category* $\mathbf{Sets}^{\mathbf{C}^{op}}$*, any object* P *is the colimit of a diagram of representable objects.*

This propositions asserts that given a functor $P : \mathbf{C}^{op} \to \mathbf{Sets}$, one can construct a small "index" category J and a diagram $A : J \to \mathbf{C}$ in $\mathbf{C}$ of type J, such that P is isomorphic to the colimit $\underset{\to J}{Lim}(\mathbf{y} \circ A)$ of the diagram $J \xrightarrow{A} \mathbf{C} \xrightarrow{\mathbf{y}} \mathbf{Sets}^{\mathbf{C}^{op}}$, obtained by composition with the Yoneda embedding.

Given P the index category J is called **category of elements** of P, denoted by $\int_{\mathbf{C}} P$ or briefly $\int P$. Its objects are all pairs (C, p) where C is an object of $\mathbf{C}$ and p is an element $p \in P(C)$. Its morphisms $(C', p') \to (C, p)$ are those morphisms $u : C' \to C$ of $\mathbf{C}$ for which $pu = p'$. This category has an evident projection functor

$$\pi_P : \int_{\mathbf{C}} P \to \mathbf{C}, \quad (C, p) \mapsto C. \tag{A.3}$$

It can be proved that [[1], p. 43]

$$P \cong Colim(\int P \xrightarrow{\pi_P} \mathbf{C} \xrightarrow{\mathbf{y}} \widehat{\mathbf{C}}). \tag{A.4}$$

Remark. *The parallel result for a sheaf category is in Proposition D.2.1, every sheaf in a sheaf category* $Sh(\mathbf{C}, J)$ *is the colimit of associated sheaves of representables.*