Exercise 2.1.6 Show that there are $\binom{n+1}{k+1}$ **injective** order preserving functions $[k] \to [n]$ (considered as elements in $\Delta[n]_k$ these are called "non-degenerate k-simplices" in $\Delta[n]$.)

A *degenerate* k-simplex in $X \in \mathcal{S}$ is an element $x \in X_k$ such that $x = \phi_* y$ for some $y \in X_n$ and non-injective $\phi\colon [k] \to [n]$.

2.1.7 For the Record: Δ Described by "Generators and Relations"

In particular, for $0 \le i \le n$ we have the maps

$$d^i\colon [n-1] \to [n], \qquad d^i(j) = \begin{cases} j & j < i \\ j+1 & i \le j \end{cases} \qquad \text{"skips } i\text{"}$$

$$s^i\colon [n+1] \to [n], \qquad s^i(j) = \begin{cases} j & j \le i \\ j-1 & i < j \end{cases} \qquad \text{"hits } i \text{ twice".}$$

Every map in Δ has a factorization in terms of these maps. Let $\phi \in \Delta([n],[m])$. Let $\{i_1 < i_2 < \cdots < i_k\} = [m] - im(\phi)$, and $\{j_1 < j_2 < \cdots < j_l\} = \{j \in [n] | \phi(j) = \phi(j+1)\}$. Then

$$\phi(j) = d^{i_k} d^{i_{k-1}} \cdots d^{i_1} s^{j_1} s^{j_2} \cdots s^{j_l}(j)\ .$$

This factorization is unique, and hence we could describe Δ as being generated by the maps d^i and s^i subject to the "cosimplicial identities" :

$$d^j d^i = d^i d^{j-1} \quad \text{for } i < j$$

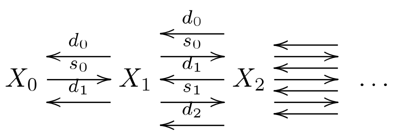
$$s^j s^i = s^{i-1} s^j \quad \text{for } i > j$$

and

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ d^{i-1} s^j & \text{for } i > j+1 \end{cases}\ .$$

If X is a simplicial set, we let X_n be the image of $[n]$, and for a map $\phi \in \Delta$ we will often write ϕ^* for $X(\phi)$. For the particular maps d^i and s^i, we write simply d_i and s_i for $X(d^i)$ and $X(s^i)$, and call them *face* and *degeneracy maps*. Note that the face and degeneracy maps satisfy the "simplicial identities" which are the duals of the cosimplicial identities.

Hence a simplicial set is often defined in the literature to be a sequence of sets X_n and maps d_i and s_i



satisfying the simplicial identities.

[DLOR07] Dundas, B. I., Levine, M., Østvær, P. A., Röndigs, O., & Voevodsky, V. (2007). Motivic homotopy theory: lectures at a summer school in Nordfjordeid, Norway, August 2002. Springer Science & Business Media.