Two examples of (co)limits as (co)equalisers

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This page is from the post Calculate (co)limits as (co)equalisers (two examples).

1. The connected component of a simplicial set

Let $S_\bullet$ be a simplicial set. Then the component map $u : S_\bullet \to \underline{\pi_0(S_\bullet)}_\bullet$ [Tag 00GP] exhibits $\pi_0(S_\bullet)$ as the colimit of the diagram $\Delta^{\mathrm{op}} \to \mathrm{Set}$ determined by $S_\bullet$ [Tag 00GR].

Proposition 1.1.6.22 [Tag 00GT] The component map $u : S_\bullet \to \underline{\pi_0(S_\bullet)}_\bullet$ [Tag 00GP] exhibits $\pi_0(S_\bullet)$ as the coequalizer of the face maps $d_0, d_1 : S_1 \rightrightarrows S_0$.

Proof. Clearly, there is a unique map $f_1 : \mathrm{coeq}(S_1 \rightrightarrows S_0) \longrightarrow \lim S_\bullet$ since the limit equalises the two face maps.

For any $\sigma : S_0 \to C$ that coequalises d_0 and d_1, we can form a cone from $S_\bullet$ to C by defining the map $S_k \to C$ to be σd_0^k as follow:

$$\begin{array}{ccccccc} & S_2 & \overset{d_0,\,d_1,\,d_2}{\Rrightarrow} & S_1 & \overset{d_0,\,d_1}{\rightrightarrows} & S_0 \\ \cdots & \big\downarrow{\scriptstyle \sigma d_0 d_0} & & \big\downarrow{\scriptstyle \sigma d_0} & & \big\downarrow{\scriptstyle \sigma} \\ & C & = & C & = & C. \end{array}$$

To show that this is a cone, we apply the simplicial identities [see Tag 000G] . Since every $S_k \to S_j$ can be decomposed into d_i and s_i, it suffices to show the commutativity of d_i's and s_i's. We finish the proof by noting the following identities obtained by repeatedly applying the simplicial identities:

$$\sigma d_0^k d_i = \sigma d_0^{k-1} d_{i-1} d_0 = \cdots = \sigma d_0^{k+1} \tag{1}$$

$$\sigma d_0^k s_i = \sigma d_0^{k-1} s_{i-1} d_0 = \cdots = \sigma d_0 s_0 d_0^{k-1} = \sigma d_0^{k-1} \tag{2}$$

for $i > 0$ (the case $i = 0$ is immediate). Since it forms a cone, by universal property there is a unique map $f_2 : \lim S_\bullet \longrightarrow \mathrm{coeq}(S_1 \rightrightarrows S_0)$ whose compositions with f_1 on both sides are the identities. □

Proof from Tag 00GV (Supplemented details). The idea is to use the bijection

$$\mathrm{Hom}_{\mathrm{Set}}(\pi_0(S_\bullet), J) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \underline{J}_\bullet)$$

for the adjunction pair: the connected component functor π_0 and the constant simplicial set functor. But the rest of details I fill in here still uses the simplicial identities.

To show a map $S_0 \to I$ factorises uniquely as

$$S_0 \xrightarrow{u_0} \pi_0(S_\bullet) \to I$$

is to show that $\underline{S_0}_\bullet \to \underline{I}_\bullet$ factorises uniquely as

$$\underline{S_0}_\bullet \to S_\bullet \to \underline{I}_\bullet$$

by adjunction correspondence.

This is equivalent to the assertion that there is a unique map of simplicial sets $F : S_\bullet \to \underline{I}_\bullet$ which coincides with f on simplicies of degree zero. Let σ be an n-simplex of $S_\bullet$ identified as a map of simplicial sets $\sigma : \Delta^n \to S_\bullet$. Consider the image $\sigma(i : [0] \to [n])$ in S_0 where i is the map with $i(0) = i$. Note that i factorises as $i = d^n d^{n-1} \dots \hat{d^i} \dots d^0$ and we want to show that $f\sigma(i) = f\sigma(j)$ for all $0 \le i \le j \le n$ so that we can define the desired unique map $F : S_\bullet \to \underline{I}_\bullet$ as $F(\sigma) = f(\sigma(i))$.

The identity $f\sigma(i) = f\sigma(j)$ follows from the commutativity of the cone we showed in my last proof. Or we can proceed directly as follow:

$$f\sigma(i) = fd_0d_1 \dots \hat{d_i} \dots d_n = fd_0^i d_{i+1} \dots d_n = fd_1 d_0^i d_{i+2} \dots d_n \tag{3}$$

$$= fd_0^{i+1} d_{i+2} \dots d_n \tag{4}$$

$$= \dots = fd_0^n \tag{5}$$

Note that line (3) is by repeatedly applying the simplicial identity $d_i d_j = d_{j-1} d_i$ for $0 \le i < j \le n$ and (4) is by the condition $fd_0 = fd_1$. □

2. The sheaf condition

Recall that a Grothendieck topology on a category C assigns each object $c \in C$ a collection of sieves $\mathrm{Cov}(c)$ satisfying some axioms.

In my post Categorical descriptions for glueing sheaves and schemes, I mentioned in the proof of Lemma 6.33.2 that for an open $V \subset X$ and a covering sieve $J_V \in \mathrm{Cov}(V)$, the sheaf condition for a sheaf $\mathcal{F}$ is just the limit over J_V:

$$\mathcal{F}(V) = \varprojlim_{(V_i \to V) \in J_V} \mathcal{F}(V_i).$$

This is the same as the equaliser usually presented in the sheaf condition:

$\mathcal{F}(V)$ is the equaliser of

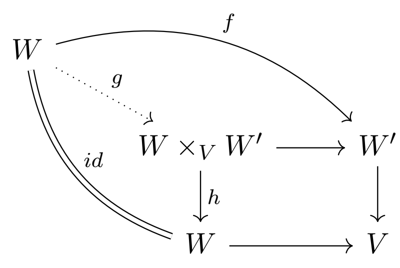
$$\prod_{W_i \in J_V} \mathcal{F}(W_i) \rightrightarrows \prod_{W_i, W_j \in J_V} \mathcal{F}(W_i \times_V M_j) \tag{6}$$

for a covering sieve J_V.

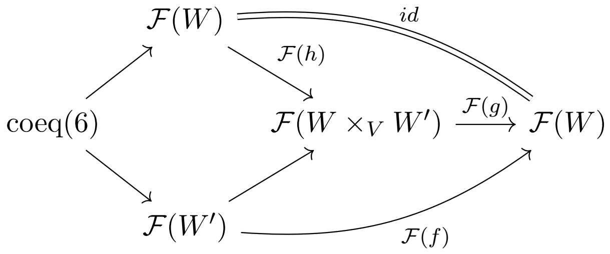
Proof. First there is a unique map from $\lim_{J_V} \mathcal{F}(V_i)$ to the coequaliser of (6) since the coequaliser diagram is part of the limit diagram.

Now we show that the map $\mathrm{coeq}(6) \to \mathcal{F}(W_i)$ given by the product map forms a cone hence there is also a unique map from $\mathrm{coeq}(6) \to \lim_{J_V} \mathcal{F}V_i$ whose compositions with the map above on both sides give the identities.

To show that it forms a cone, for any $f : W \to W'$ in J_V, there is a unique map $g : W \to W \times_V W'$ by the universal property as follow:



This gives the following commutative diagram and the commutativity condition for the cone follows from the outer square.



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