Two examples of (co)limits as (co)equalisers

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This page is from the post Calculate (co)limits as (co)equalisers (two examples).

1. The connected component of a simplicial set

Let $S_\bullet$ be a simplicial set. Then the component map $u : S_\bullet \to \pi_0(S_\bullet)$ [Tag 00GP] exhibits $\pi_0(S_\bullet)$ as the colimit of the diagram $\Delta^{op} \to \text{Set}$ determined by $S_\bullet$ [Tag 00GR].

**Proposition 1.1.6.22** [Tag 00GT] The component map $u : S_\bullet \to \pi_0(S_\bullet)$ [Tag 00GP] exhibits $\pi_0(S_\bullet)$ as the coequalizer of the face maps $d_0, d_1 : S_1 \to S_0$.

**Proof.** Clearly, there is a unique map $f_1 : \text{coeq}(S_1 \to S_0) \to \lim S_\bullet$ since the limit equalises the two face maps.

For any $\sigma : S_0 \to C$ that coequalises $d_0$ and $d_1$, we can form a cone from $S_\bullet$ to $C$ by defining the map $S_k \to C$ to be $\sigma d_k^0$ as follow:

$\begin{align*}
S_2 & \xrightarrow{d_0} S_1 \xrightarrow{d_0} S_0 \\
\cdots & \downarrow \sigma d_0 d_0 \downarrow \sigma d_0 \downarrow \sigma \\
C & \xrightarrow{\sigma} C \\
\end{align*}$

To show that this is a cone, we apply the simplicial identities [see Tag 000G]. Since every $S_k \to S_j$ can be decomposed into $d_i$ and $s_i$, it suffices to show the commutativity of $d_i$’s and $s_i$’s. We finish the proof by noting the following identities obtained by repeatedly applying the simplicial identities:

\begin{align*}
\sigma d_k^i d_i &= \sigma d_0^{k-1} d_{i-1} d_0 = \cdots = \sigma d_0^{k+1} & (1) \\
\sigma d_k^i s_i &= \sigma d_0^{k-1} s_{i-1} d_0 = \cdots = \sigma d_0 s_0 d_0^{k-1} = \sigma d_0^{k-1} & (2)
\end{align*}

for $i > 0$ (the case $i = 0$ is immediate). Since it forms a cone, by universal property there is a unique map $f_2 : \lim S_\bullet \to \text{coeq}(S_1 \to S_0)$ whose compositions with $f_1$ on both sides are the identities. \qed

**Proof from Tag 00GV** (Supplemented details). The idea is to use the bijection

$\text{Hom}_{\text{Set}}(\pi_0(S_\bullet), J) \to \text{Hom}_{\Delta}(S_\bullet, I_\bullet)$

for the adjunction pair: the connected component functor $\pi_0$ and the constant simplicial set functor. But the rest of details I fill in here still uses the simplicial identities.

To show a map $S_0 \to I$ factorises uniquely as

$\begin{align*}
S_0 & \xrightarrow{u_0} \pi_0(S_\bullet) \to I \\
\end{align*}$

is to show that $S_0 \to I$ factorises uniquely as

$\begin{align*}
\overline{S}_0 & \to \overline{I} \\
\end{align*}$

by adjunction correspondence.
This is equivalent to the assertion that there is a unique map of simplicial sets \( F : S_\bullet \to I_\bullet \) which coincides with \( f \) on simplicies of degree zero. Let \( \sigma \) be an \( n \)-simplex of \( S_\bullet \) identified as a map of simplicial sets \( \sigma : \Delta^n \to S_\bullet \). Consider the image \( \sigma([0] \to [n]) \) in \( S_0 \) where \( i \) is the map with \( i(0) = i \). Note that \( i \) factorises as \( i = d^n d^{n-1} \ldots d^i \ldots d^0 \) and we want to show that \( f \sigma(i) = f \sigma(j) \) for all \( 0 \leq i \leq j \leq n \) so that we can define the desired unique map \( F : S_\bullet \to I_\bullet \) as \( F(\sigma) = f(\sigma(i)) \).

The identity \( f \sigma(i) = f \sigma(j) \) follows from the commutativity of the cone we showed in my last proof. Or we can proceed directly as follow:

\[
\begin{align*}
    f \sigma(i) &= f d_0 d_1 \ldots d_i \ldots d_n = f d_0 d_0 d_i+1 \ldots d_n = f d_0 d_0 d_1 d_2 \ldots d_n. \\
    &= f d_0 d_0 d_i+1 \ldots d_n \\
    &= \cdots = f d_0
\end{align*}
\]

Note that line (3) is by repeatedly applying the simplicial identity \( d_i d_j = d_j - 1 d_i \) for \( 0 \leq i < j \leq n \) and (4) is by the condition \( f d_0 = f d_1 \).

2. The sheaf condition

Recall that a Grothendieck topology on a category \( C \) assigns each object \( c \in C \) a collection of sieves \( \text{Cov}(c) \) satisfying some axioms.

In my post Categorical descriptions for glueing sheaves and schemes, I mentioned in the proof of Lemma 6.33.2 that for an open \( V \subset X \) and a covering sieve \( J_V \in \text{Cov}(V) \), the sheaf condition for a sheaf \( \mathcal{F} \) is just the limit over \( J_V \):

\[
\mathcal{F}(V) = \lim_{\leftarrow (V_i \to V) \in J_V} \mathcal{F}(V_i).
\]

This is the same as the equaliser usually presented in the sheaf condition:

\[
\mathcal{F}(V) \text{ is the equaliser of } \prod_{W_i \in J_V} \mathcal{F}(W_i) \Rightarrow \prod_{W_i, W_j \in J_V} \mathcal{F}(W_i \times_V M_j)
\]

for a covering sieve \( J_V \).

**Proof.** First there is a unique map from \( \lim_{J_V} \mathcal{F}(V_i) \) to the coequaliser of (6) since the coequaliser diagram is part of the limit diagram.

Now we show that the map \( \text{coeq}(6) \to \mathcal{F}(W_i) \) given by the product map forms a cone hence there is also a unique map from \( \text{coeq}(6) \to \lim_{J_V} \mathcal{F}V_i \) whose compositions with the map above on both sides give the identities.

To show that it forms a cone, for any \( f : W \to W' \) in \( J_V \), there is a unique map \( g : W \to W \times_V W' \) by the universal property as follow:

\[
\begin{array}{ccc}
  W & \xrightarrow{g} & W' \\
  \downarrow{id} & & \downarrow{h} \\
  W \times_V W' & \xrightarrow{f} & W' \\
  \downarrow{h} & & \downarrow{h} \\
  W & \xrightarrow{g(id)} & V
\end{array}
\]
This gives the following commutative diagram and the commutativity condition for the cone follows from the outer square.