K Theory of Vector Bundles

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Abstract

We prove Serre-Swan theorem which establishes equivalences between the categories of topological vector bundles over a compact Hausdorff space X, the category of finitely generated projective $C(X)$-modules and the categories of algebraic vector bundles of finite rank over the affine scheme $\operatorname{Spec} C(X)$. This theorem connects different objects of interest in K-theory. We also introduce some ideas on the construction of K-groups for a Banach category and in particular for compact topological spaces and Banach algebras.

1 Introduction

K-theory was introduced by A. Grothendieck to formalise work on Riemann-Roch Theorem [BS58]. For each projective algebraic variety, Grothendieck constructed a group from the category of coherent algebraic sheaves. Atiyah and Hirzebruch considered a topological analogue defined for any compact space X and defined the "topological K-theory", a contravariant functor $K(-)$ on the category of vector bundles on X [AH61]. This K group can be extended to a generalised cohomology theory $K^n(X, Y)$ for compact spaces X and closed subspace Y of X. On the other hand, we have algebraic K-theory which study the K-group $K(A)$ for interesting rings A and it can also be extended to $K_n(A)$ for $n \in \mathbb{Z}$.

An important bridge between these objects is Serre-Swan Theorem [[Swa62],[Ser55]] which shows that the category of vector bundles on a compact Hausdorff space X is equivalent to the category of finitely generated projective modules over $C(X)$; and the latter is also equivalent to the category of algebraic vector bundles on the affine scheme $\operatorname{Spec} C(X)$. In section 2.1, we prove Swan's theorem from Swan's original paper [Swa62] which shows the equivalence for compact Hausdorff space X. This result can also be generalised to arbitrary X without the requirement of being compact Hausdorff [Vas86]. Section A contains a few properties about inner product for paracompact spaces that we will use for the proof of Swan's Theorem.

In section 2.2, we motivate the definition of finite rank locally free sheaf (also called algebraic vector bundle) on a ringed space as a generalisation of a topological vector bundle. Then we show the equivalence of the category of algebraic vector bundles on the affine scheme $\operatorname{Spec} A$ and the category of finitely generated projective A-modules [Ser55]. A key algebraic input is the result that for A a finitely generated module over a Noetherian ring R, the projective dimension of A is equal to the supremum of the projective dimension of $A_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-module for all maximal ideal $\mathfrak{m} \subset A$. This result was left as an exercise in [CE99] and we finished the proof for this exercise in Section B.

After Atiyah and Hirzebruch defined topological K-theory from the category of vector bundles on a compact space X, the equivalence in Serre-Swan Theorem motivates a similar definition of K-group $K(A)$ of a ring with unit by replacing vector bundles with finitely generated projective A-modules [Section 3]. This K-group is a covariant functor on the category of rings. By the equivalence of Swan's Theorem, we have $K(X) \cong K(A)$ for $A = C_k(X)$ the ring of continuous k-valued function on X.

Similar to what Atiyah and Hirzebruch did for vector bundles, one would like to define new functors $K_n(A), n \in \mathbb{Z}$ which extends the definition $K_0(A) = K(A)$. This is more difficult for general ring. In Section 4.1, we start with the case for A a Banach algebra over the base field $k = \mathbb{R}$ or $\mathbb{C}$ and have a look at some properties for the covariant K-groups of Banach algebras. Then following the approach in [Kar08], in Section 4.2 and 4.3 we consider K-groups as a contravariant functor on a Banach category. In particular, in Section 4.4 we define the groups $K^{-n}(X, Y)$ for pairs (X, Y) with Y a closed subspace of a compact space X. Finally in Section 4.5, this definition is generalised to $K^{-n}(X, Y; A)$ for a Banach algebra A which gives back to $K^{-n}_{\mathbb{R}}(X, Y)$ or $K^{-n}_{\mathbb{C}}(X, Y)$ when taking A to be either $\mathbb{R}$ or $\mathbb{C}$. Moreover, we obtain contravariant K-groups $K^{-n}(A) := K^{-n}(\{pt\}, \emptyset; A)$ of Banach algebras A which turn out to be isomorphic to $K_n(A)$ in Section 4.1.

2 Characterisation of Vector Bundles

2.1 Swan's Theorem [Swa62]

Let k denote either the field of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. (In later sections on K-theory, we mostly care about the case where k is either $\mathbb{R}$ or $\mathbb{C}$.) Let $C(X) = C_k(X)$ be the ring of continuous k-valued functions on X. If ξ is a k-vector bundle over X and let $\Gamma(\xi)$ be the set of all sections of ξ over X. If $s_1, s_2 \in \Gamma(\xi)$, define $(s_1 + s_2)(x) = s_1(x) + s_2(x)$. If $s \in \Gamma(\xi)$ and $a \in C(X)$, define $(as)(x) = a(x)s(x)$. With these operations, $\Gamma(\xi)$ becomes a $C(X)$-module. Moreover, Γ is an additive functor from the category of k-vector bundles to the category of $C(X)$-modules.

First we show that if X is normal, then the functor Γ is fully faithful, that is, Γ gives an isomorphism $\mathrm{Hom}(\xi, \eta) \cong \mathrm{Hom}_{C(X)}(\Gamma(\xi), \Gamma(\eta))$.

Note that we have the following equivalent condition for a space to be normal.

Lemma 2.1 (Exer.I.5.6, [Bre13])**.** *A (Hausdorff) space X is normal if and only if for any sets U open and C closed with $C \subset U$ there is an open set V with $C \subset V \subset \bar{V} \subset U$.*

Proof. Suppose X is normal and $C \subset U$ for C closed and U open. Let W_1, W_2 be two disjoint open sets separating C and U^c. Since $W_1 \subset W_2^c$ and W_2^c is closed, we have $C \subset W_1 \subset \bar{W}_1 \subset W_2^c \subset U$.

Conversely, let F, G be two disjoint closed sets in X. Then we have $G \subset F^c$, there is an open set U with $G \subset U \subset \bar{U} \subset F^c$. On the other hand, we have $F \subset \bar{U}^c$, thus there is an open set V such that $F \subset V \subset \bar{V} \subset \bar{U}^c \subset U^c$. So U, V are two disjoint open sets separating F and G. □

We also have the Urysohn's lemma [I.10.2, [Bre13]]: if X is normal and $F \subset U$ where F is closed and U is open, then there is a map $f : X \to [0, 1]$ which is 0 on F and 1 on $X \setminus U$.

Lemma 2.2 (Lemma 3, [Swa62])**.** *Let X be normal. Let U be a neighborhood of x, and let s be a section of a vector bundle ξ over U. Then there is a section s' of ξ over X such that s' and s agree in some neighborhood of x.*

Proof. Let $U' \subset U$ be an open set containg x. Note that a space is a T_1-space if and only if every singleton set is closed. Applying Lemma 2.1 to $\{x\} \subset U'$, there is an open set W such that $\{x\} \subset W \subset \bar{W} \subset U'$. Applying Lemma 2.1 again to $\bar{W} \subset U'$, there is an open set V such that $\bar{W} \subset V \subset \bar{V} \subset U'$.

Let ω be a real valued function on X such that $\omega|_{\bar{W}} = 1$, $\omega|_{X \setminus V} = 0$. Let $s'(y) = \omega(y)s(y)$ if $y \in U$ and $s'(y) = 0$ if $y \notin U$. Then s' is the desired section of ξ over X. □

For any vector bundle ξ over X and $x \in X$, there is a neighborhood U of x and sections $s_1, \ldots s_n$ of ξ over U such that for any $y \in U$, $s_1(y), \ldots s_n(y)$ form a k-base for F_y. We say that $s_1, \ldots, s_n$ form a **local base** at x. Any section s of ξ over U can be written as $s(y) = \Sigma_{i=1}^n a_i(y) s_i(y)$ where $a_i(y) \in k$. Note that s is continuous if and only if each a_i is. By Lemma 2.2, we can extend a local base to some global sections, hence we have the following:

Corollary 2.1 (Corollary 1,[Swa62]). *Let X be normal. For any $x \in X$ there are elements $s_1, \ldots s_n \in \Gamma(\xi)$ which form a local base at x.*

Lemma 2.3 (Corollary 3, [Swa62]). *Let I_x be the 2-sided ideal of $C(X)$ consisting of all $a \in C(X)$ with $a(x) = 0$. Then $\Gamma(\xi)/I_x\Gamma(\xi) \cong E(\xi)_x$ with isomorphism given by $s \mapsto s(x)$.*

Proof. Let $f : \Gamma(\xi)/I_x\Gamma(\xi) \to E(\xi)_x$ be the map given by $s \mapsto s(x)$ which is clearly a homomorphism. Given $p \in E(\xi)_x$, there is a section s over a neighborhood U of p with $s(x) = p$. By lemma 2.2, we can extend this section to a global section $s' \in \Gamma(\xi)$ with $s'(x) = p$. Thus f is surjective.

To show injectivity, suppose $s(x) = 0$ and let $s_1, \ldots, s_n \in \Gamma(\xi)$ be a local base at x which exists by Corollary 2.1. Let $s(y) = \Sigma_{i=1}^n b_i(y) s_i(y)$ near x, $b_i(y) \in k$. Applying Lemma 2.2 to the trivial bundle $X \times k$, we can extend the local k-valued functions b_i to global functions $a_i \in C(X)$ such that a_i and b_i agree in a neighborhood U of x. Then $s' := s - \Sigma_{i=1}^n a_i s_i$ vanishes in a neighborhood U of x. Let V be an open set containing x such that $\bar{V} \subset U$ (by normality of X, Lemma 2.1). Let $a \in C(X)$ be 0 at x and 1 on $X \setminus V$. Then $s = as' + \Sigma_{i=1}^n a_i s_i$. Since $a(x) = 0$ and $a_i(x) = b_i(x) = 0$, we have that $s \in I_x\Gamma(X)$. □

Theorem 2.1 (Theorem 1, [Swa62]). *Let X be normal. Given a $C(X)$-map $F : \Gamma(\xi) \to \Gamma(\eta)$, there is a unique k-vector bundle map $f : \xi \to \eta$ such that $F = \Gamma(f)$.*

Proof. Given $F : \Gamma(\xi) \to \Gamma(\eta)$, F induces a map $f_x : \Gamma(\xi)/I_x\Gamma(\xi) \to \Gamma(\eta)/I_x\Gamma(\eta)$ which gives a map $f : E(\xi) \to E(\eta)$ by Lemma 2.3. This map is k-linear on fibres. If $s \in \Gamma(\xi)$, then $(fs)(x) = f_x s(x) = (F(s))(x)$, thus $F = \Gamma(f)$. It remains to show that it's continuous.

Let $s_1, \ldots, s_m \in \Gamma(\xi)$ be a local base at x on some neighborhood U of x and $p : E(\xi) \to X$ the projection of the vector bundle ξ. Then for any $y \in p^{-1}(U)$, $y = \Sigma_{i=1}^m a_i(p(y)) s_i(p(y))$ where the a_i are continuous k-valued function on U. Then $f(y) = \Sigma_{i=1}^m a_i(p(y)) f s_i(p(y))$. Since $fs_i = F(s_i)$, fs_i is a continuous section of η. Now all terms in the sum are continuous in y and since continuity of a function is a local property, we conclude that s is continuous.

Now we show the uniqueness. Suppose $f, g : \xi \to \eta$ are two k-vector bundle maps such that $\Gamma(f) = \Gamma(g)$. Given $y \in E(\xi)_x$, there is a local section s over a neighborhood of x with $s(x) = y$. We can extend it to a global section $s' \in \Gamma(X)$ with $s'(x) = y$ by Lemma 2.2. Now $f(y) = fs'(x) = (\Gamma(f)s')(x) = (\Gamma(g)s')(x) = g(y)$, that is, $f = g$. □

Next, we will show that if X is compact Hausdorff, then the functor Γ from the category of k-vector bundles over X to the category of finitely generated projective $C(X)$-modules is essentially surjective.

Note that if ξ is the trivial bundle $E(\xi) = X \times k^n \to X$, then $\Gamma(\xi)$ is obviously a free $C(X)$-module with n generators.

Proposition 2.1. *If X is compact Hausdorff and ξ is any k-vector bundle over X, then $\Gamma(\xi)$ is a finitely generated projective $C(X)$-module.*

Proof. First, we show that there exists a trivial vector bundle ζ and an epimorphism $f : \zeta \to \xi$ such that ξ is a direct summand of ζ.

For each $x \in X$, choose a set of sections $s_{x,1}, \ldots, s_{x,k_x} \in \Gamma(\xi)$ which form a local base over some neighborhood U_x of x. Since X is compact, a finite number of the U_x cover X. So there are a finite number of sections (note that we can extend local sections to global sections by Lemma 2.2) $s_1, \ldots s_n \in \Gamma(\xi)$ such that $s_1(x), \ldots, s_n(x)$ span $E(\xi)_x$ for every x. Let ζ be the trivial bundle with $E(\xi) = X \times k^n$. Then $\Gamma(\zeta)$ is a free $C(X)$-module in n generators, denoted by $e_1, \ldots, e_n$. Since $\Gamma(\zeta)$ is free, a homomorphism from $\Gamma(\xi)$ to anothe $C(X)$-module is determined by its values on the free basis for $\Gamma(\zeta)$. Define $F : \Gamma(\zeta) \to \Gamma(\xi)$ by $F(e_i) = s_i$. By Theorem 2.1, this corresponds to a map $f : \zeta \to \xi$. Since $fe_i = s_i$ and $s_i(x) \in \operatorname{im} f$, f is onto.

$$\begin{array}{ccc} E(\zeta) & \xrightarrow{\quad f \quad} & E(\xi) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

$$\begin{array}{ccc} \Gamma(\zeta) & \xrightarrow{\quad F \quad} & \Gamma(\xi) \\ e_i & \longmapsto & s_i \end{array}$$

Now let $\eta = \ker f$. By proposition A.1, η is a subbundle since $\operatorname{im} f = \xi$ is a subbundle. By Proposition A.3, there is a subbunble ξ' with $\xi' \cong \xi$ such that $\zeta = \eta \oplus \xi'$. Since Γ is an additive functor, $\Gamma(\xi)$ is a direct summand of $\Gamma(\xi)$ which is a finitely generated free $C(X)$-module. So we conclude that $\Gamma(\xi)$ is a finitely generated projective $C(X)$-module. □

Theorem 2.2 (Theorem 2, [Swa62])**.** *Let X be compact Hausdorff. Then a $C(X)$-module P is isomorphic to a module of the form $\Gamma(\xi)$ if and only if P is finitely generated and projective.*

Proof. The "only if" direction follows from Proposition 2.1. For the "if" direction, suppose that P is finitely generated and projective. Then P is a direct summand of a finitely generated free $C(X)$-module F. Therefore, there is an idempotent endomorphism $g : F \to F$ (i.e. $g^2 = g$) with $P \cong \operatorname{im} g$. Now $F = \Gamma(\zeta)$ where ζ is a trivial k-vector bundle. By Theorem 2.1, $g = \Gamma(f)$ for some unique $f : \zeta \to \zeta$.

Now we show that $\ker f$ is a subbundle of ζ so that by Proposition A.3, $\operatorname{im} f$ is also a subbundle and $\zeta = \ker f \oplus \operatorname{im} f$ and $P \cong \operatorname{im} \Gamma(f) = \Gamma(\operatorname{im} f)$.

Let $\xi = \operatorname{im} f$ and $\eta = \ker f$. To show that $\xi = \operatorname{im} f$ is a subbundle, by Proposition A.1, it suffices to show that $\dim_k F_x(\xi)$ is locally constant. We first show the following claim.

Claim. Let $f : \xi_1 \to \xi_2$ be a map of vector bundles. If $\dim F_x(\operatorname{im} f) = k$, then $\dim F_y(\operatorname{im} f) \geq k$ for all y in some neighborhood of x.

Proof of Claim. First, note that if $t_1, \ldots, t_k$ are sections of a vector bundle ξ over a neighborhood U of x such that $t_1(x), \ldots, t_k(x)$ are linearly independent. Then there is a neighborhood V of x such that $t_1(y), \ldots, t_k(y)$ are linearly independent [Lemma 1, [Swa62]]. To see this, let $s_1, \ldots, s_n$ be a local base at x. Let $t_i(y) = \Sigma a_{ij}(y) s_j(y)$. At $y = x$, all $k \times k$ minors of the matrix $(a_{ij}(x))$ must be nonsingular. Therefore, continuity of (a_{ij}) implies that all the $k \times k$ minors of the $(a_{ij}(y))$ must be nonsingular for y in some neighborhood of x. (For $K = \mathbb{R}$ or $\mathbb{C}$, this follows by taking determinant. For the field of quaternions, one first replace the matrix with a $4k \times 4k$ one. Then we know that the nonvanishing of the real determinant of the new matrix is equivalent to the nonsingularity of the original one.)

Now let $s_1, \ldots, s_m$ be a local base for ξ_1. Let k be the dimension of the fibre $E(\operatorname{im} f)_x$ of $\operatorname{im} f$. Without loss of generality, we may assume that $fs_1(x), \ldots, fs_k(x)$ span $E(\operatorname{im} f)_x$ and so are linearly independent. From above there is a some neighborhood U of x such that $fs_1(y), \ldots fs_k(y)$ are linearly independent for all $y \in U$. So $\dim E(\operatorname{im} f)_y \geq k$. ■

Since $g^2 = g$, by Theorem 2.1, we have $f^2 = f$. So $\eta = \ker f = \operatorname{im}(1-f)$ and $E(\zeta)_x = E(\xi)_x \oplus E(\eta)_x$. Suppose $\dim E(\xi)_x = h$, $\dim E(\eta)_x = k$. By the claim applied to f and $1-f$, we have $\dim E(\xi)_y \geq h$, $\dim E(\eta)_y \geq k$ for all y in some neighborhood of x. Then

$$\dim E(\xi)_y + \dim E(\eta)_y = \dim E(\zeta)_y = h + k$$

is a constant. Thus $\dim E(\zeta)_x$ is locally constant. □

Remark. *The process above shows that the kernel of an idempotent endomorphism $f : \zeta \to \zeta$ is a subbundle and ζ splits into $\zeta = \ker(f) \oplus \operatorname{im}(f) = \ker(f) \oplus \ker(1-f)$. Generally, this is true for a* ***pseudo-abelian category*** *which is an additive category such that for any morphism p with $p^2 = p$, the kernel of p exists. In a pseudo-abelian category $\mathcal{C}$, for any $p : E \to E$ such that $p^2 = p$, the object E splits into the direct sum $E = \ker(p) \oplus \ker(1-p)$ [I.6.9, [Kar08]].*

Since a compact Hausdorff space is normal, by Theorem 2.1 and 2.2, we conclude that the category of vector bundles over a compact Hausdorff space X is equivalent to the category of finitely generated projective $C(X)$-modules under the additive functor Γ.

2.2 Serre's Theorem

2.2.1 Vector bundles and locally free sheaves

First, we give some motivations for locally free sheaves which can be seen as a generalisation of vector bundles in algebraic geometry. Recall that a **rank n (real) vector bundle on a manifold** M is a map $\pi : V \to M$ such that each fibre $\pi^{-1}(x)$ is an n-dimensional vector space for each $x \in M$ and that for every $x \in M$, there is an open neighborhood U and a homeomorphisms

$$\Phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

over U so that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\cong} & U \times \mathbb{R}^n \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & G & \end{array}$$

commutes, and such that for every $x \in M$, $\phi_x : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ is an isomorphism of vector spaces. An isomorphism as above is called a **trivialization over** U.

Given trivializations $\phi_1 : U_1 \times \mathbb{R}^n \to \pi^{-1}(U_1)$ and $\phi_2 : U_2 \times \mathbb{R}^n \to \pi^{-1}(U_2)$, the two trivializations must be related by a function $T_{12} : U_1 \cap U_2 \to \mathrm{Gl}_n(\mathbb{R})$ such that $\phi_1(u, x) = \phi_2(u, T_{12}(u)x)$. Moreover, if $\{U_i\}$ is an open cover of M, given trivializations on each U_i, the functions $\{T_{ij}\}$ must satisfy the **cocycle condition**:

$$T_{jk}|_{U_i \cap U_j \cap U_k} \circ T_{ij}|_{U_i \cap U_j \cap U_k} = T_{ik}|_{U_i \cap U_j \cap U_k}. \tag{1}$$

In particular, the condition implies that $T_{ij} = T_{ji}^{-1}$. The data $\{T_{ij}\}$ are called the **transition functions** for the trivialization.

Conversely, a vector bundle can be recovered up to unique isomorphism from the data of a cover $\{U_i\}$ and transitions functions T_{ij} by glueing construction [P.370, [Vak17]].

Suppose we are given a continuous map $\mu : Y \to X$. Define the "sections of μ" as the presheaf $\mathcal{F}$ which assigns each open set U of X the set of continuous maps $s : U \to Y$ such that $\mu \circ s = id|_U$. It's easy to see that $\mathcal{F}$ forms a sheaf by directly checking the definition, we call this sheaf **the sheaf of sections**.

Recall that a **sheaf of $\mathcal{O}_X$-modules** (or simply an **$\mathcal{O}_X$-module**) is a sheaf $\mathcal{F}$ on X such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module, and for each inclusion $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Let M be a differentiable manifold and $\mathcal{O}_M$ be the sheaf of differentiable functions, that is, for each open set $U \subset M$, $\mathcal{O}_M(U)$ is the ring of differentiable functions $U \to \mathbb{R}$. Then $(M, \mathcal{O}_M)$ is a ringed space. Given a rank n vector bundle $E \to M$. The sheaf of sections $\mathcal{F}$ of E is an $\mathcal{O}_M$-module: given any open set U, we can multiply a section over U by a function on U and get another section. Given a trivialization over U, the sections over U are naturally identified with functions $U \to \mathbb{R}^n$. Thus given a cover $\{U_i\}$ and trivializations over each U_i, we have an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$

$$\begin{array}{c} U \times \mathbb{R}^n \\ \pi \big\downarrow \quad \big\uparrow \text{n-tuple of functions} \\ U \end{array} \cdot$$

This motivates the following definition of locally free sheaves.

Definition 2.1 (P109, [Har13]). *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is* ***free*** *if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is* ***locally free*** *if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module.*

Now we say that the sheaf of sections $\mathcal{F}$ as above is a locally free sheaf of rank n on the ringed space $(M, \mathcal{O}_M)$.

Note that the open cover $\{U_i\}$ of X in the definition determines transition functions $T_{ij} \in \mathrm{Gl}_n(\mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (1). This in turn determines a locally free sheaf.

Definition 2.2. *An* ***algebraic vector bundle*** *over a ringed space $(X, \mathcal{O}_X)$ is a locally free $\mathcal{O}_X$-module whose rank is finite at every point. We will write* $\mathbf{VB}(X)$ *or* $\mathbf{VB}(X, \mathcal{O}_X)$ *for the category of vector bundles on $(X, \mathcal{O}_X)$.*

The morphisms in $\mathbf{VB}(X)$ are just morphisms of $\mathcal{O}_X$-modules. Since the direct sum of locally free modules is locally free, $\mathbf{VB}(X)$ is locally free.

Definition 2.3 (P111, [Har13]). *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is* ***quasi-coherent*** *if X can be covered by open affine subsets $U_i = \operatorname{spec} A_i$ such that for each i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We say that $\mathcal{F}$ is coherent if furthermore each M_i can be taken to be a finitely generated A_i module.*

[See P.110, [Har13] for the definition of ***the sheaf*** $\widetilde{M}$ ***associated*** *to an A-module M on* $\operatorname{Spec} A$. *]*

We say a property P enjoyed by some affine open subsets of a scheme X is **affine local** if the following hold:

(i) If an affine open subset $\operatorname{Spec} A \subset X$ has property P then for any $f \in A$, $\operatorname{Spec} A_f \subset X$ does too.

(ii) If $(f_1, \ldots, f_n) = A$, and $\operatorname{Spec} A_{f_i} \subset X$ has P for all i, then so does $\operatorname{Spec} A \subset X$.

Suppose that $X = \bigcup_{i \in I} \operatorname{Spec} A_i$ where $\operatorname{Spec} A_i$ has property P which is affine local, then every affine open subset of X has P too [Affine Communication Lemma, [Vak17]].

Proposition 2.2 (13.2.1, [Vak17]). *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Suppose P is the property of affine open subschemes* $\operatorname{Spec} A$ *of X that $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M. Then P is affine local.*

Proof. See Lemma 5.3 and Proposition 5.4, [Har13]. $\square$

Thus we see that a sheaf of $\mathcal{O}_X$-module is quasicoherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, there is an A-module M such that $\mathcal{F}|_U \cong \widetilde{M}$.

Proposition 2.3 (Exercise 13.3.D, [Vak17]). *An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for every distinguished inclusion $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, there is an isomorphism $\Gamma(\operatorname{Spec} A, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} A_f, \mathcal{F})$.*

Proof. Let $\phi : \Gamma(\operatorname{Spec} A, \mathcal{F}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ be the restriction map. The source of ϕ is an A-module, and the target is an A_f-module, by the universal property of localization, ϕ naturally factors as:

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} A, \mathcal{F}) & \xrightarrow{\quad\phi\quad} & \Gamma(\operatorname{Spec} A_f, \mathcal{F}) \\ & {\scriptstyle \otimes_A A_f}\searrow \qquad \nearrow{\scriptstyle \alpha} & \\ & \Gamma(\operatorname{Spec} A, \mathcal{F})_f & \end{array} .$$

If $\mathcal{F}$ is quasicoherent, then $\mathcal{F}|_{\operatorname{Spec} A} = \widetilde{M}$ for some A-module M. Then by definition of $\widetilde{M}$, $\Gamma(\operatorname{Spec} A_f, \mathcal{F}) = M_f = \Gamma(A, \mathcal{F})_f$. For the converse, let M be the A-module $M := \Gamma(\operatorname{Spec} A, \mathcal{F})$, the isomorphism shows that $\Gamma(\operatorname{Spec} A_f, \mathcal{F}) = M_f$. Since the distinguished open sets form a base, and a sheaf defined on the base of X extends uniquely to a sheaf of X up to unique isomorphism, we have $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$. $\square$

Thus a quasicoherent sheaf is equivalent to the data for each affine open subset $\operatorname{Spec} A$ an A-module such that the module over a distinguished open set $\operatorname{Spec} A_f$ is given by localizing the module over $\operatorname{Spec} A$. In particular, for an affine scheme X. We have the following.

Corollary 2.2 (Cor.II.5.5,[Har13]/ 49, [Ser55]). *Let A be a ring and let $X = \operatorname{Spec} A$. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of A-module and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. If A is Noetherian, the same functor also gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules.*

Now we are interested in the category of finite rank locally free sheaves which is a subcategory of the category of coherent sheaves. Thus we want to restrict the functor described above to finite rank locally free sheaves for which we have the following characterisation.

Proposition 2.4 (50, Proposition 4, [Ser55]). *Let A be a Noetherian ring and $X = \operatorname{Spec} A$. Let M be a finitely generated A-module. Then M is projective if and only if for all $x \in X = \operatorname{Spec} A$, the $\mathcal{O}_x$-module $\mathcal{O}_x \otimes_A M$ is free.*

Proof. Note that in this case $\mathcal{O}_x$ is just the local ring A_x. If M is projective, then $\mathcal{O}_x \otimes_A M = M_x$ is a projective $\mathcal{O}_x$-module. Since $\mathcal{O}_x$ is a local ring, any (finitely generated) projective module over $\mathcal{O}_x$ is free [Theorem 2.5, [Mat89]].

Conversely, if all $\mathcal{O}_x \otimes_A M$ are free, then in particular they are projective. Let pd_R denote the projective dimension for R-module. Then $\operatorname{pd}_{A_x}(\mathcal{O}_x \otimes_A M) = 0$. By Proposition B.3, $\operatorname{pd}_R M \leq \sup \operatorname{pd}_{A_\mathfrak{m}}(\mathcal{O}_\mathfrak{m} \otimes_A M) = 0$ where $\mathfrak{m}$ ranges over all maximal ideals of A, thus M is projective. $\square$

Combining Corollary 2.2 and Proposition 2.4, we establish the following equivalence of categories:

Theorem 2.3 ([Ser55])**.** *let R be a Noetherian ring (in particular, the coordinate ring of an affine variety over a field) and $X = \operatorname{Spec} R$, then the category of finitely generated projective R-modules is equivalent to the category of finite rank locally free sheaves of $\mathcal{O}_X$-modules (algebraic vector bundles).*

Remark (Characterisation of Quasicoherent Sheaves on $\operatorname{Proj} S_\bullet$)**.** *So far we have seem a nice characterisation of quasicoherent/coherent/finite rank locally free sheaves on an affine scheme. Let $S_\bullet$ be a finitely generated graded algebra generated in degree 1, and M a graded $S_\bullet$-module, denote the associated sheaf to M by $\widetilde{M}$ [P.116, [Har13]]. We define a functor $\Gamma_\bullet$ from the category $\mathrm{Qcoh}_{\operatorname{Proj} S_\bullet}$ of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ to the category of graded $S_\bullet$-modules as follow*

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_\bullet, \mathcal{F}(n)),$$

where Γ is the global section functor and $\mathcal{F}(n)$ is the n-twisted sheaf of $\mathcal{F}$.

We call a graded $S_\bullet$-module N saturated if the induced map $N \to \Gamma_\bullet(\widetilde{N})$ is an isomorphism. Indeed, the functors $\widetilde{-}$ and $\Gamma_\bullet(-)$ are an adjoint pair between $\mathrm{Qcoh}_{\operatorname{Proj} S_\bullet}$ and the category of graded $S_\bullet$-modules. They induce an equivalence between $\mathrm{Qcoh}_{\operatorname{Proj} S_\bullet}$ and the full subcategory consisting of all saturated graded $S_\bullet$-modules [P.417, [Vak17]].

3 The Grothendieck group K

Consider an abelian monoid M, that is, a set with a composition law which satisfies all the properties of an abelian group except the requirement of existence of an inverse.

Definition 3.1. *The* ***group completion*** *of an abelian monoid M is an abelian group $S(M)$ and a homomorphism of monoids $s : M \to S(M)$ which satisfies the following universal property:*

For any abelian group G and any homomorphism of the underlying monoids $f : M \to G$, there is a unique group homomorphism $\tilde{f} : S(M) \to G$ which makes the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\quad s \quad} & S(M) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & G & \end{array}$$

We now give some explicit constructions of the group completion. By the universal property of group completion, they are all isomorphic.

Construction 1. Consider the free abelian group $\mathcal{F}(M)$ with basis the elements $[m]$ of M. Then $S(M)$ is the quotient of $\mathcal{F}(M)$ by the subgroup generated by linear combinations of the form $[m+n] - [m] - [n]$ and s takes $m \in M$ to its equivalence class $[m]$.

Construction 2. Consider the product $M \times M$ and form the quotient by the equivalence relation

$$(m, n) \sim (m', n') \iff \exists p \in M, m + n' + p = n + m' + p,$$

or by

$$(m, n) \sim (m', n') \iff \exists p, q \in M, (m, n) + (p, p) = (m', n') + (p', q').$$

In either case, the quotient monoid is a group and $s(m)$ is the equivalence class of the pair $(m, 0)$.

Note that in all of these constructions, every element of $S(M)$ is of the form $[m] - [n]$ for some $m, n \in M$.

3.1 K of an additive category

Consider an additive category $\mathcal{C}$. The set $\Phi(\mathcal{C})$ of isomorphism classes of objects in $\mathcal{C}$ forms a monoid. The group completion $K(\Phi(\mathcal{C}))$ of $\Phi(\mathcal{C})$ is called the **Grothendieck group** of $\mathcal{C}$ and is written as $K(\mathcal{C})$. Any additive functor $\phi : \mathcal{C} \to \mathcal{C}'$ naturally induces a monoid homomorphism $\Phi(\mathcal{C}) \to \Phi(\mathcal{C}')$ denoted by ϕ_*.

Example 1 (K of a ring with unit). Let A be a ring with unit. Let $\mathcal{P}(A)$ be the category with objects finitely generated projective right A-modules and morphisms A-linear maps. Let $K(A)$ be the Grothendieck group of $\mathcal{P}(A)$. Note that this gives a covariant functor of the ring A by extension of scalars.

Example 2 (K of topological vector bundles). Let k denote either the field of either the real numbers, the complex numbers or quaternions. Let X be a compact topological space and $\mathcal{E}_k(X)$ be the category of k-vector bundles over X. Let $K(X)$ be the Grothendieck group of $\mathcal{E}_k(X)$.

For compact Hausdorff space X, let $C_k(X)$ be the ring of continuous k-valued function on X, then $\mathcal{P}(C(X))$ is equivalent to $\mathcal{E}_k(X)$ [Theorem 2.2]. So they have isomorphic K-groups.

Example 3 (K of algebraic vector bundles). Let $(X, \mathcal{O}_X)$ be a locally ringed space and $\mathbf{VB}(X)$ be the category of algebraic vector bundles [Definition 2.2] on X. Let $K_0(X)$ be the Grothendieck group of $\mathbf{VB}(X)$.

When X is affine, $X = \operatorname{Spec} R$, $\mathcal{P}(R)$ is equivalent to $\mathbf{VB}(X)$ [Theorem 2.3]. So they have isomorphic K-groups.

We would like to extend this definition and try to define $K_n, n \in \mathbb{Z}$ such that they form a generalised (co)homology theory.

4 The K Groups as a Generalised (Co)homology Theory

From now on, we take the base field k to be either $\mathbb{C}$ or $\mathbb{R}$.

4.1 K_n of Banach Algebras

If A is an algebra and $||\cdot||$ is a norm on A satisfying $||ab|| \leq ||a|| \cdot ||b||$, for all $a, b \in A$, then $||\cdot||$ is called an algebra norm and $(A, ||\cdot||)$ is called a **normed algebra**. A complete normed algebra is called a **Banach algebra**. If a Banach algebra A is unital, we denote the identity element as e and assume that $||e|| = 1$.

In particular we will work with modules over complex Banach algebras which correspond to complex vector bundles. We could also consider the real case while we will have an 8-periodicity instead of a 2-periodicity in Bott Periodicity Theorem.

Let X be a compact Hausdorff space and $A = C(X)$. Then with respect to the point-wise multiplication of functions, A as a commutative unital algebra with the sup norm $||f||_\infty = \sup_{x \in X} |f(x)|$ is a Banach algebra.

We can extend the definition of $K(A)$ to not necessarily unital algebras A by adding a unit to A. Consider the vector space $\mathbb{C} \oplus A = \bar{A}$ provided with the “twisted” multiplication defined by [Kar05]

$$(\lambda, a) \cdot (\lambda', a') = (\lambda\lambda', \lambda a' + a\lambda' + aa').$$

The algebra now has a unit $(1, 0)$. There is an obvious augmentation $\bar{A} \to \mathbb{C}$ and we define $K(A)$ as the kernel of the induced map $K(\bar{A}) \to K(\mathbb{C}) = \mathbb{Z}$. Note that if A already has a unit, we recover the previous definition $K(A)$.

Let $A_n = A(\mathbb{R}^n)$ be the Banach algebra of continuous function from $\mathbb{R}^n$ to A which vanishes when $x \to \infty$. For $n \in \mathbb{N}$, define $K_n(A)$ as $K(A_n)$ [Kar05].

Example 4 ([Kar05]). Let $A = C(X)$ with X locally compact. Let X^+ be the one point compactification of X and let $Y = S^n(X^+)$ be the n-th suspension of X^+. Then $K(Y)$ is isomorphic to $K_n(A) \oplus \mathbb{Z}$. To see this, note that $C(Y)$ is isomorphic to $C(X \times \mathbb{R}^n)$. In particular $K(S^n)$ is isomorphic to $K_n(k) \oplus \mathbb{Z}$ ($k = \mathbb{R}$ or $\mathbb{C}$).

Moreover, we have the following theorem whose proof can be extracted from [Kar08].

Theorem 4.1 ([Kar08])**.** *The functor $K_n(-), n \in \mathbb{N}$ on the category of Banach algebras satisfies the following properties:*

1) Exactness: for any exact sequence of Banach algebras (where A'' has the quotient norm and A' the induced norm)

$$0 \to A' \to A \to A'' \to 0,$$

there is an exact sequence of K-groups

$$\cdots \to K_{n+1}(A) \to K_{n+1}(A'') \to K_n(A') \to K_n(A) \to K_n(A'') \to \ldots.$$

2) Homotopy: $K_n(A(I)) \cong K_n(A)$ where $A(I)$ is the ring of continuous functions on the unit interval I with values in A.

3) Normalization: $K_0(A)$ is the group $K(A)$ defined before.

A continuous bilinear pairing of Banach algebras

$$A \times C \to B$$

induces a "cup-product"

$$K_i(A) \times K_j(C) \to K_{i+j}(B)$$

which has associative and graded commutative properties [II.4.1, [Kar08]]. Take $C = k$, $A = B$, then the cup product induces a map

$$K_i(A) \otimes K_j(A) \to K_{i+j}(A).$$

We now have the Bott Periodicity in the setting of Banach algebra by which we can define $K_{-n}(A)$ for $n \in \mathbb{N}$.

Theorem 4.2 ([Kar03], [Woo66])**.** *1. Let A be a complex Banach algebra. Then $K_2(\mathbb{C}) \cong \mathbb{Z}$ and the cup-product with a generator u_2 induces an isomorphism $\beta_{\mathbb{C}}$: $K_n(A) \to K_{n+2}(A)$.*

2. Let A be a real Banach algebra. Then $K_8(\mathbb{R}) \cong \mathbb{Z}$ and the cup-product with a generator u_8 induces an isomorphism $\beta_{\mathbb{R}} : K_n(A) \to K_{n+8}(A)$.

We have an isomorphism $K_n(A) \cong \operatorname{colim}_r \pi_{n-1}(\mathrm{GL}_r(A)) = \pi_{n-1}(\mathrm{GL}(A))$ ([Kar05], [Kar08]), this theorem implies that the periodicity of the homotopy group of the infinite general linear group for a Banach algebra

Corollary 4.1. *1. If A is a complex Banach algebra, we have [AB64]*

$$\pi_i(\mathrm{GL}(A)) \cong \pi_{i+2}(\mathrm{GL}(A)) \text{ and } \pi_1(\mathrm{GL}(A)) \cong K(A).$$

2. If A is a real Banach algebra, we have [Kar03], [Woo66]

$$\pi_i(\mathrm{GL}(A)) \cong \pi_{i+8}(\mathrm{GL}(A)) \text{ and } \pi_7(\mathrm{GL}(A)) \cong K(A).$$

4.2 The Grothendieck Group of a Functor

More generally, we can define K-groups for Banach categories. We also want to define a *relative* K-group associated to a functor [II. [Kar08]].

Let $\mathcal{C}$ be an additive category. A **Banach category** on $\mathcal{C}$ is given by a Banach space structure on all the groups $\hom_{\mathcal{C}}(E, F)$, where E and F run through the objects of $\mathcal{C}$. Moreover, the map

$$\hom_{\mathcal{C}}(E, F) \times \hom_{\mathcal{C}}(F, G) \to \hom_{\mathcal{C}}(E, G)$$

given by the composition of morphisms, is bilinear and continuous. A **Banach category** is an additive category provided with a Banach structure [II.2.2, [Kar08]].

Definition 4.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories, and $\phi : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Then ϕ is called **quasi-surjective** if every object of $\mathcal{C}'$ is a direct factor of an object $\phi(E)$ for some $E \in \mathrm{Ob}(\mathcal{C})$.*

*If $\mathcal{C}$ and $\mathcal{C}'$ are Banach categories, the functor ϕ is called a **Banach functor** if the map $\hom_{\mathcal{C}}(E, F) \to \hom_{\mathcal{C}'}(\phi(E), \phi(F))$ is linear and continuous.*

Example 5 (II.2.7, [Kar08]). Let $\phi : \mathcal{E}(X) \to \mathcal{E}(Y)$ be the functor between the categories of vector bundles on X and Y defined by $\phi(E) = E|_Y$, where Y is a closed subspace of the compact space X. Then ϕ is a Banach functor which is full and quasi-surjective by Proposition A.3.

Example 6 (II.2.8, [Kar08]). Let $f : Y \to X$ be a continuous map. Then the induced functor $f^* : \mathcal{E}(X) \to \mathcal{E}(Y)$ is a quasi-surjective Banach functor.

Example 7 (II.2.9, [Kar08]). Let A, B be Banach algebras, and $u : A \to B$ a continuous homomorphism. Then u induces a functor $u_* : \mathcal{P}(A) \to \mathcal{P}(B)$, defined by extension of scalars $E \mapsto E \otimes_A B$. u_* is a quasi-surjective Banach functor.

Let $\phi : \mathcal{C} \to \mathcal{C}'$ be a quasi-surjective Banach functor. Let $\Gamma(\phi)$ denote the set of triples (E, F, α) where E and F are objects of $\mathcal{C}$ and $\alpha : \phi(E) \to \phi(F)$ is an isomorphism. Two triples (E, F, α) and (E', F', α') are called **isomorphic** if there exist isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the following diagram commutes [II.2.13, [Kar08]].

$$\begin{array}{ccc} \phi(E) & \xrightarrow{\alpha} & \phi(F) \\ {\scriptstyle \phi(f)}\downarrow & & \downarrow{\scriptstyle \phi(g)} \\ \phi(E') & \xrightarrow[\alpha']{} & \phi(F') \end{array}$$

A triple (E, F, α) is called **elementary** if $E = F$ and if α is homotopic to $\mathrm{Id}_{\phi(E)}$ within the automorphism of $\phi(E)$. And we define the sum of two triples (E, F, α) and (E', F', α') to be $(E \oplus E', F \oplus F', \alpha \oplus \alpha')$ [II.2.13, [Kar08]].

Define $K(\phi)$ to be the quotient of $\Gamma(\phi)$ by the following equivalence relation:

$\sigma \sim \sigma' \iff$ there exist elementary τ and τ' such that $\sigma + \tau$ is isomorphic to $\sigma' + \tau'$.

The sum of triples provide the set $K(\phi)$ with a monoid structure. Let $d(E, F, \alpha)$ denote the class of (E, F, α) in the monoid $K(\phi)$. Note that $d(E, F, \alpha) = 0$ if and only if there exist objects G and H of $\mathcal{C}$, isomorphisms $u : E \oplus G \to H$ and $v : F \oplus G \to H$, such that $\phi(v) \cdot (\alpha \oplus \mathrm{Id}_{\phi(G)}) \cdot \phi(u^{-1})$ is homotopic to $\mathrm{Id}_{\phi(H)}$ within the automorphisms of $\phi(H)$ [Kar08].

Now we define the K-group $K(X, Y)$ as $K(\phi)$. The group $K(X, Y)$ depends functorially on the pair (X, Y) [II.2.33, [Kar08]]. More precisely, recall that a morphism between pairs (X, Y)

and (X', Y') is a continuous map $f : X \to X'$ such that $f(Y) \subset Y'$. Such a morphism induces a commutative diagram of categories

$$\begin{array}{ccc} \mathcal{E}(X) & \longrightarrow & \mathcal{E}(Y) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f|_Y^*} \\ \mathcal{E}(X') & \longrightarrow & \mathcal{E}(Y') \end{array}$$

Hence it induces a morphism f^* between $K(X, Y)$ and $K(X', Y')$ given by the formula

$$f^*(d(E', F', \alpha')) = d(f^*(E), f^*(F), f_1^*(\alpha')).$$

Now consider the quotient space X/Y. If Y is non-empty, then X/Y is the compact space obtained by identifying Y with a single point, denoted by y. Note that X/Y is the one point compactification of the locally compact space $X \setminus Y$ [II.2.34, [Kar08]].

K is a contravariant functor on the category of compact spaces [II.1.12, [Kar08]], the projection of X onto a point P induces a homomorphism $\alpha : \mathbb{Z} \cong K(P)$ whose cokernel is denoted by $\widetilde{K}(X)$ and called the **reduced K-theory** of X. When we want to specify the base field $k = \mathbb{R}$ or $\mathbb{C}$, we write $\widetilde{K}_{\mathbb{R}}(X)$ or $\widetilde{K}_{\mathbb{C}}(X)$.

Theorem 4.3 (Excision, 2.35, [Kar08])**.** *The projection $\pi : X \to X/Y$ induces an isomorphism*

$$\pi^* : K(X/Y, \{y\}) \to K(X, Y).$$

From the above theorem, we have that $K(X/Y, \{y\}) \cong \ker(K(X/Y) \to K(\{y\})) \cong \widetilde{K}(X/Y)$.

4.3 The Group K^{-1} of a Banach Category

Let $\mathcal{C}$ ba a Banach category. Consider the set of pairs (E, α), where E is an object of $\mathcal{C}$ and α is an automorphism of E. Two pairs (E, α) and (E', α') are isomorphic if there is an isomorphism $h : E \to E'$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ E & \xrightarrow[h]{} & E' \end{array}$$

The sum of two pairs (E_0, α_0) and (E_1, α_1) is defined to be $(E_0 \oplus E_1, \alpha_0 \oplus \alpha_1)$. A pair (E, α) is **elementary** if α is homotopic to Id_E within the automorphisms of E.

Define $K^{-1}(\mathcal{C})$ to be the quotient of the set of pairs (E, α) by the equivalence relation: $\sigma \sim \sigma' \iff$ there exist elementary τ and τ' such that $\sigma + \tau$ is isomorphic to $\sigma' + \tau'$.

Example 8 (II.3.15, [Kar08]). Let A be a Banach algebra. Then $K^{-1}(\mathcal{P}(A)) \cong \pi_0(\mathrm{GL}(A))$.

For the category of k-vector bundles over a compact space X, we have the following:

Theorem 4.4 (3.17, [Kar08])**.** *Let X be a compact space and let $[X, \mathrm{GL}(k)] \cong \operatorname{inj\,lim}[X, \mathrm{GL}_n(k)]$, $k = \mathbb{R}$ or $\mathbb{C}$ be the set of homotopy classes of continuous maps from X to $\mathrm{GL}(k)$. There is an isomorphism $u : \operatorname{inj\,lim}[X, \mathrm{GL}_n(k)] \to K^{-1}(X) = K^{-1}(\mathcal{E}_k(X))$.*

Corollary 4.2 (Cor 3.19, [Kar08])**.** *Let $O = \operatorname{inj\,lim} O(n)$ (resp. $U = \operatorname{inj\,lim} U(n)$)) be the infinite orthogonal group (resp. the infinite unitary group). Then we have a natural isomorphisms*

$$[X, O] \cong \operatorname{inj\,lim}[X, O(n)] \xrightarrow{\cong} K_{\mathbb{R}}^{-1}(X)$$

and

$$[X, U] \cong \operatorname{inj\,lim}[X, U(n)] \xrightarrow{\cong} K_{\mathbb{C}}^{-1}(X).$$

4.4 The group $K^{-n}(X, Y)$ for a pair of spaces

Definition 4.2 (4.11, [Kar08])**.** *If Y is a closed subspace of a locally compact space X, define $K^{-n}(X, Y)$ to be $K((X \setminus Y) \times \mathbb{R}^n)$.*

This definition agrees up to isomorphism with the definition of $K^{-1}(X) = K^{-1}(X, \emptyset)$.

Proposition 4.1 (4.12,[Kar08])**.** *If X is a compact space and if Y is a closed subspace, we have natural isomorphisms $K^{-n}(X, Y) \cong \widetilde{K}(S^n(X/Y)) \cong K(X \times B^n, X \times S^{n-1} \cup Y \times B^n)$.*

Proof. We have the homeomorphisms

$$X \times B^n \setminus (X \times S^{n-1} \cup Y \times B^n) \cong (X \setminus Y) \times (B^n \setminus S^{n-1}) \cong (X \setminus Y) \times \mathbb{R}^n.$$

Hence,

$$K(X \times B^n, X \times S^{n-1} \cup Y \times B^n) \cong K((X \setminus Y) \times (B^n \setminus S^{n-1})) \cong K((X \setminus Y) \times \mathbb{R}^n).$$

Moreover, $S^n(X/Y) \cong B^n/S^{n-1} \wedge X/Y \cong X \times B^n/(X \times S^{n-1} \cup Y \times B^n)$. By Excision 4.3, we have the last isomorphism $\widetilde{K}(S^n(X/Y)) \cong K(X \times B^n, X \times S^{n-1} \cup Y \times B^n)$. □

Similar to the case for $K_n(-)$ on the category of Banach algebras, by the following theorem (Bott Periodicity), one can extend the definition of $K^n(X, Y)$ for positive n.

Theorem 4.5 (III.1.3, [Kar08])**.** *Let X be a locally compact space and Y be a closed subspace. Then there is an isomorphism $K_{\mathbb{C}}^{-n}(X, Y) \cong K_{\mathbb{C}}^{-n}(X \times B^2, X \times S^{-1} \cup Y \times B^2) = K_{\mathbb{C}}^{n-2}(X, Y)$.*

Remark. *For $k = \mathbb{R}$, the isomorphism becomes 8-periodicity: $K_{\mathbb{R}}^{-n}(X, Y) \cong K_{\mathbb{R}}^{n-8}(X, Y)$.*

The groups $K^n(X, Y), n \in \mathbb{Z}$ have the following properties which implies that it's a generalised cohomology theory.

Theorem 4.6 (II.3.1, [Kar08])**.** *The groups $K^n(X, Y)$, $n \in \mathbb{Z}$ satisfy the following axioms:*

1) Exactness: There exist natural transformations

$$\partial^{n-1} : K^{n-1}(Y) \to K^n(X, Y).$$

such that the sequence

$$K^{n-1}(X) \xrightarrow{j^*} K^{n-1}(Y) \xrightarrow{\partial^{n-1}} K^n(X, Y) \xrightarrow{i^*} K^n(X) \xrightarrow{j^*} K^n(Y)$$

is exact, where in general $K^n(Z) = K^n(Z, \emptyset)$ and the maps j^* and i^* are induced by inclusions $(Y, \emptyset) \subset (X, \emptyset)$ and $(X, \emptyset) \subset (X, Y)$ respectively.

2) Homotopy. If $f_0, f_1 : (X, Y) \to (X', Y')$ are homotopic continuous maps between pairs, they induce the same homomorphisms $f_0^* = f_1^* : K^n(X', Y') \to K^n(X, Y)$.

3) Excision. The projection $(X, Y) \to (X/Y, \{y\})$ induces an isomorphism $K^n(X/Y, \{y\}) \xrightarrow{\cong} K^n(X, Y)$.

4) Normalization. The functor $K^0(X) = K^0(X, \emptyset)$ is the functor $K(X)$ constructed in Example 2.

4.5 The group $K^{-n}(X, Y; A)$ for a Banach algebra A

We now come back to Banach algebras and generalise the definition of K-groups $K_n(X, Y)$ to allow "coefficient algebra" besides $\mathbb{R}$ and $\mathbb{C}$. Also we will obtain a contravariant functors $K^n(-)$ on the category of Banach algebras. These construction is based on our previous K-groups of a Banach category.

Suppose A is a Banach algebra, and X is a compact space. Let $A(X)$ denote the ring of continuous functions on X with values in A. If Y is a closed subspace of X, we have a Banach functor $\phi : \mathcal{P}(A(X)) \to \mathcal{P}(A(Y))$ associated with the ring map $A(X) \to A(Y)$. We define $K(X, Y; A)$ as the Grothendieck group of the functor ϕ [Exercise 6.14, [Kar08]].

By Swan's Theorem 2.2, we have the equivalence of categories $\mathcal{P}(A(X)) \cong \mathcal{E}_k(X)$ for $A = \mathbb{R}$ or $\mathbb{C}$, thus we recover the previous definition for real and complex K-groups, $K_{\mathbb{R}}(X, Y) \cong K(X, Y; \mathbb{R})$ and $K_{\mathbb{C}}(X, Y) \cong K(X, Y; \mathbb{C})$.

Now we define $K^{-n}(X, Y; A) = K(X \times B^n, X \times S^{n-1} \cup Y \times B^n; A)$. Similar to the techniques used in II.4, [Kar08] where the particular case for the Banach category $\mathcal{E}_k(X)$ is considered, one can show the exact sequence

$$K^{-n-1}(X; A) \to K^{-n-1}(Y; A) \to K^{-n}(X, Y; A) \to K^{-n}(X; A) \to K^{-n}(Y; A).$$

Define $K^{-n}(A)$ to be $K^{-n}(\{p\}; A)$ where p is a point. If B is a Banach algebra without unit element, we define $K^{-n}(B)$ as $\ker[K^{-n}(B^+) \to K^{-n}(\mathbb{R})]$, where B^+ denotes the algebra augmented by $\mathbb{R}$.

If

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence of Banach algebras, then there is an induced long exact sequence

$$K^{-n-1}(A) \to K^{-n-1}(A'') \to K^{-n}(A') \to K^{-n}(A) \to K^{-n}(A'').$$

Let $A(X)$ be the Banach algebra of continuous functions with values in A. Then $K^{-n}(X; A) \cong K^{-n}(A(X))$.

Note that for the case X is one point, $\mathcal{P}(A(X))$ becomes $\mathcal{P}(A)$ and $K^0(A)$ recovers the previous definition of the Grothendieck group $K(A)$ of A. Moreover, $K^{-n-1}(A) \cong \pi_n(\mathrm{GL}(A))$ [Exer.II.6.14(d), [Kar08]] thus $K^n(A)$ turns out to be isomorphic to the $K_n(A)$ defined in Section 4.1.

A Inner product for paracompact space

In general, it is not true that the kernel or image of a map of vector bundles is a vector bundle. For example, let $X = I$ be the unit interval and ξ the trivial bundle $I \times k$. Let $f : \xi \to \xi$ be given by $f(x, y) = (x, xy)$. Then the image of f has a fibre of dimension 1 everywhere except at $x = 0$ where the fibre is zero. But the dimension of a vector bundle is locally constant hence the dimension of any vector bundle over I must be constant since I is connected. So $\operatorname{im} f$ can not be a vector bundle over I. A similar argument shows that $\ker f$ is not a vector bundle either. However we have the following equivalent conditions.

Proposition A.1 (Proposition 1,[Swa62])**.** *Let $f : \xi \to \eta$ be a map of vector bundles. Then the following statements are equivalent:*

(1) $\operatorname{im} f$ is a subbundle of η;

(2) $\ker f$ is a subbundle of ξ;

(3) the dimensions of the fibres of $\operatorname{im} f$ are locally constant;

(4) the dimensions of the fibres of $\ker f$ are locally constant.

The proof is omitted here and it follows a similar argument as the proof of the claim in Theorem 2.2.

Definition A.1. *An inner product on a vector bundle* $\xi : E \to B$ *is a map* $\langle , \rangle : E(\xi) \times E(\xi) \to K$ *which restricts in each fibre to an inner product, a positive definite symmetric bilinear form.*

Proposition A.2 (Proposition 1.2, [Hat03]/Lemma 2, [Swa62])**.** *Let* X *be paracompact, then any* k*-vector bundle* ξ *over* X *has an inner product.*

Proof. Let $\{U_\alpha\}$ be a locally finite covering of X such that $p^{-1}(U_\alpha) = U_\alpha \times k^{n_\alpha}$. First construct an inner product $\langle , \rangle_{\alpha,x}$ on each $p^{-1}(U_\alpha)$. Let $\{\phi_\alpha\}$ be a real partition of unity on X for the covering $\{U_\alpha\}$. Define $\langle u, v\rangle_x = \Sigma_\alpha \phi_\alpha(x)\langle u, v\rangle_{\alpha,x}$. □

Proposition A.3 (Proposition 2, [Swa62])**.** *If* X *is paracompact, any subbundle* η *of a vector bundle* ξ *is a direct summand.*

Proof. Since X is paracompact, there exists an inner product on k-vector bundles over X by Proposition A.2. This defines at each $x \in X$ the orthogonal complement $E(\eta)_x^\perp \subset E(\xi)_x$ of $E(\eta)_x \subset E(\xi)_x$.

Define a map $f : \xi \to \eta$ given at each fibre by the projection via the inner product $f_x : E(\xi)_x \to E(\eta)_x$. Clearly $(\ker f)_x = E(\eta)_x^\perp$ is locally constant since $E(\eta)_x$ and $E(\xi)_x$ are, so by Proposition A.1, $\ker f$ is a subbundle and $\xi = \eta \oplus \ker f$. □

B Projective and Weak Dimension

This section contains some details of the result quoted in the proof of Serre' Theorem 2.3. The proofs follow the exercises in [CE99].

Recall that the **projective dimension** for an A-module M is defined to be the minimal length of a projective resolution if there exists one of finite length

$$0 \to P_n \to P_{n-1} \to \dots P_1 \to P_0 \to M \to 0;$$

otherwise it's defined to be ∞. We denote the projective dimension of an A-module M by $\mathrm{pd}_A(M)$.

For M a left R-module, so that $T(-) = - \otimes_R M$ is a right exact functor, the Tor-groups is defined to be the left derived functor $\mathrm{Tor}_n^R(-, M) := (L_n T)(-)$.

There is the following characterisation of projective dimension.

Proposition B.1 (Lemma 4.1.6, [Wei95])**.** *The following are equivalent for a right* R*-module* A*:*

1. $\mathrm{pd}(A) \leq d$.
2. $\mathrm{Ext}_R^n(A, B) = 0$ *for all* $n > d$ *and all* R*-modules* B*.*
3. $\mathrm{Ext}_R^{d+1}(A, B) = 0$ *for all* R*-modules* B*.*
4. *Given an exact sequence* $0 \to M_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ *with* P_i*'s projective, then the module* M_d *is also projective.*

Definition B.1 (VI.Exer. 3, [CE99])**.** *The* ***weak dimension*** *or* ***Tor-dimension*** *of a left* R*-module* A *is defined to be the highest integer* n *such that* $Tor_n^R(C, A) \neq 0$ *for some right* R*-module* C*, denoted by* $\mathrm{w.dim}_R A$*.*

Definition B.2 (tag 08XJ,[Sta20])**.** *Let* $\mathcal{A}$ *be an abelian category. An injection* $A \subset B$ *of* $\mathcal{A}$ *is* ***essential****, or we say that* B *is an* ***essential extension*** *of* A*, if every nonzero subobject* $B' \subset B$ *has nonzero intersection with* A*.*

Let R *be a ring. An injection* $M \to I$ *of* R*-modules is said to be an* ***injective hull*** *if* I *is an injective* R*-module and* $M \to I$ *is an essential extension.*

Lemma B.1 (Tag 08Y3, [Sta20]). *Let R be a ring. Any R-module has an injective hull.*

Proof. Let M be an R-module. The category of R-modules have enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. Order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset, then $\bigcup E_\alpha$ is an essential extension of M too. Apply Zorn's Lemma, we have a maximal element $E \in \mathcal{S}$. By Tag 08XS, [Sta20], we see that an R-module is injective if and only if every essential extension is trivial. Thus the maximality of E implies that E is injective, $M \subset E$ is an essential injection. □

Proposition B.1 (Exer.VI.3, [CE99]). *If R is left Noetherian and A is a finitely generated R-module, then*

$$\mathrm{w.dim}_R A = \mathrm{pd}_R A.$$

Proof. Since a projective resolution is used to calculate the Tor groups, $\mathrm{w.dim}_R A \leq \mathrm{pd}_R A$ always holds for any R-module A.

Suppose $\mathrm{pd}_R(A) = n$, then $\mathrm{Ext}^n_R(A, B) \neq 0$ for some R-module B. Let C be an injective hull of $\mathrm{Ext}^R_n(A, B)$, then by Proposition VI.5.3, [CE99], $\mathrm{Tor}^n_R(\mathrm{Hom}_R(B, C), A) \cong \mathrm{Hom}_R(\mathrm{Ext}^n_R(A, B), C) \neq 0$. Thus $\mathrm{w.dim}_R A \geq n$. □

Remark. *Weak dimension can be characterised by flat resolution:*

$\mathrm{w.dim}_R A \leq n \iff$ *Given a flat resolution* $0 \to M_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to A \to 0$ *with F_i's flat, then the module M_d is also flat.*

Proposition B.2 (Exer.VII.10, [CE99]). *Let S be a multiplicatively closed subset of R, then for any R-modules A,B,*

$$\mathrm{Tor}^{S^{-1}R}_n(S^{-1}A, S^{-1}B) \cong S^{-1}(\mathrm{Tor}^R_n(A, B)).$$

Proof. Let $X_\bullet$ be a projective resolution of A, then $S^{-1}X_\bullet$ is a projective resolution of $S^{-1}A$. We have

$$\begin{aligned} S^{-1}(\mathrm{Tor}^R_n(A, B)) &= S^{-1}(H_n(X_\bullet \otimes_R B)) = H_n(S^{-1}(X_\bullet \otimes_R B)) \\ &= H_n(S^{-1}X_\bullet \otimes_{S^{-1}R} S^{-1}B) = \mathrm{Tor}^{S^{-1}R}_n(S^{-1}A, S^{-1}B). \end{aligned}$$

□

Proposition B.3 (Exer.VII.11, [CE99]). *For any R-module A,*

$$\mathrm{w.dim}_R A = \sup \mathrm{w.dim}_{R_\mathfrak{m}} A_\mathfrak{m}$$

where the supremum is taken over all maximal ideals $\mathfrak{m} \subset R$.

If R is Noetherian and A is finitely generated, then

$$\mathrm{pd}_R A = \sup \mathrm{pd}_{R_\mathfrak{m}} A_\mathfrak{m}.$$

Proof. For any $R_\mathfrak{m}$ module B, $B_\mathfrak{m} = B$, Proposition B.2 gives that

$$\mathrm{Tor}^{R_\mathfrak{m}}_n(A_\mathfrak{m}, B) = \mathrm{Tor}^{R_\mathfrak{m}}_n(A_\mathfrak{m}, B_\mathfrak{m}) = (\mathrm{Tor}^R_n(A, B))_\mathfrak{m}.$$

Note that for any R-module C, $C = 0$ if and only if $C_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \subset R$. It follows that $\mathrm{w.dim}_R A = \sup \mathrm{w.dim}_{R_\mathfrak{m}} A_\mathfrak{m}$. The second equality follows from Proposition B.1. □

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