The homology ring of $F\psi^q$

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Here we calculate the homology ring of $F\psi^q$ which is one step in the computation of K theory of finite fields, see section 4, Mitchell.

Let $\mathbb{F}_q$ denote the field with q elements, where $q = p^d$, p a prime.

Back story

Here is Quillen's theorem on K theory of finite fields.

Theorem.

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i - 1 \\ 0, & n \textit{ even.} \end{cases} \tag{1}$$

Let $\psi^q : BU \to BU$ be the map realizing Adams operation on $\tilde{K}X$ and $\chi : BU \to BU$ be the map realizing the multiplicative inverse of BU (BU is an H-space and let m denote the multiplication).

We form the difference $\psi^q - 1$:

$$BU \xrightarrow{\Delta} BU \times BU \xrightarrow{\psi^q \times \chi} BU \times BU \xrightarrow{m} BU$$

and let $F\psi^q$ be the homotopy fibre of this map.

The homotopy groups of $F\psi^q$ are the same as in (1). This follows from the long exact sequence of the fibre sequence and that the Adams operation ψ^k on $\tilde{KU}(S^{2n})$ is multiplication by k^n.

Next we construct a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ in [Section 3, Mitchell] as follow. By Green's theoerm, for any finite group G and any representation of G over $\mathbb{F}_q$, the Brauer character is a virtual complex character, that is, the character of a virtual complex representation. This means that we could lift a representation over $\mathbb{F}_q$ to a representation over $\mathbb{C}$. Since representation is uniquely determined by its character up to isomorphism, this gives a map between representation rings $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$. Now take $G = GL_n(\mathbb{F}_q)$, consider the Brauer character χ_n of the standard representation of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. This gives a map $GL_n(\mathbb{F}_q) \to GL_n(\mathbb{C})$ hence a map $BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$. Compatibility ($\chi_n|_{GL_{n-1}} = \chi_{n-1}$) and universality of $BGL(\mathbb{F}_q)^+$ gives a map $BGL\mathbb{F}_q^+ \to BU$.

Then we show that this map composed with $\psi^q - 1$ is nullhomotopic hence induces a map to its homotopy fibre $F\psi^q$, call this map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. Both of these spaces are H-spaces and to show that θ is a homotopy equivalence, we show that it induces isomorphisms in homology groups. (This is referred to as Whitehead's thoerem for H-spaces. Another way to see this is to note that a homology isomorphism between H-spaces $f : X \to Y$ exhibits Y as a +-construction of X relative to the trivial subgroup of $\pi_1(X)$ and by universality of relative + construction [IV, Theorem 1.5, The K-Book, Weibel].)

To show that the map θ induces isomorphisms in integral homology, it suffices to show that it induces isomorphism in rational, mod p, and mod l homology for prime $(l, p) = 1$. The rational and mod p homology rings are both trivial [Section 3, Mitchell].

We want to show the isomorphism in mod l homology by starting with calculating the mod l homology ring of $F\psi^q$ in the case $l | q - 1$.

Homology ring of $F\psi^q$

Propositon [Lemma 4.5, Mitchell]. If $l|q-1$, then $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras.

Proof. Consider the fibre sequence $U \to F\psi^q \to BU$ coming from the fibre sequence $F\psi^q \to BU \to BU$. Let $L : X \mapsto X_{(l)}$ denote the localization of X away from l. Let $\beta : BU \to \Omega_0^2 BU$ denote the Bott map, then we form a diagram where $h = L(\psi^q - 1)$

$$\begin{array}{ccc} BU & \xrightarrow{\psi^q-1} & BU \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\ BU_{(l)} & \xrightarrow{h} & BU_{(l)} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ \Omega_0^2 BU_{(l)} & \xrightarrow{\Omega_0^2 h} & \Omega_0^2 BU_{(l)} \end{array}$$

Note that the Bott isomorphism $\beta : \tilde{K}X \xrightarrow{\cong} \tilde{K}(S^2 \wedge X)$ does not commute with the Adams operations. In fact, we have the formula $\psi^q(\beta a) = q\beta(\psi^q a)$ because $\beta a = b \times a$ with $b \in \tilde{K}S^2$ and ψ^q is multiplicative. Going around the right side gives $a \mapsto \beta\psi^q a - \beta a = 1/q\psi^q\beta a - \beta a$ and going around the left side gives $a \mapsto \psi^q \beta a - \beta a$. Moreover, multiplication by q is an equivalence for $BU_{(l)}$ since $(l, q) = 1$. We have that the above diagram homotopy commutes and h is equivalent to $\Omega_0^2 h$, a double-loop map. Now the fibre sequence associated to h is multiplicative meaning that h commutes with the H-space multiplication, thus the spectral sequence is a spectral sequence of Hopf algebras. Since L induces isomorphisms in mod l homologies, the same is true for $\psi^q - 1 : BU \to BU$.

Consider the fibre sequence mod l homology associated to $F\psi^q \to BU \to BU$. Since $\pi_1 BU$ is trivial, the E_2 term of this fibre sequence is $H_*U \otimes H_*BU$. We know that $H_*BU \cong \mathbb{Z}/l[c_1, c_2, \ldots]$ where $|c_i| = 2i$ the Chern classes and $H_*U \cong \mathbb{Z}/l\langle x_1, x_2, \ldots\rangle$ with $|x_i| = 2i-1$. Since the bidegree of a differential is one odd and one even, the spectral sequence collapses at E_2. Thus the mod l homology $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras. $\square$