K-theory of Finite Fields

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This is an outline of Quillen's proof for the calculation of K-theory of finite fields, originally done by Quillen in [2], see also [1] for a slightly different presentations with more background materials included.

Let $\mathbb{F}_q$ denote the field with q elements, where $q = p^d$, p a prime.

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1 Back story

Here is Quillen's theorem on K theory of finite fields. Below I'll summarize and outline the main points of the proof.

Theorem.

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i-1 \\ 0, & n \text{ even.} \end{cases} \tag{1}$$

Let $\psi^q : BU \to BU$ be the map realizing Adams operation on $\tilde{K}X$ and $\chi : BU \to BU$ be the map realizing the inverse of BU (BU is an H-space and let m denote the multiplication).

We form the difference $\psi^q - 1$:

$$BU \xrightarrow{\Delta} BU \times BU \xrightarrow{\psi^q \times \chi} BU \times BU \xrightarrow{m} BU$$

and let $F\psi^q$ be the homotopy fibre of this map.

The homotopy groups of $F\psi^q$ are the same as in (1). This follows from the long exact sequence of the fibre sequence $F\psi^q \to BU \xrightarrow{\psi^q - 1} BU$ and that the Adams operation ψ^k on $\widetilde{KU}(S^{2n})$ is multiplication by k^n.

Next we construct a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ [Section 3, Mitchell] as follow. By Green's theoerm, for any finite group G and any representation of G over $\mathbb{F}_q$, the Brauer character is a virtual complex character, that is, the character of a virtual complex representation. This means that we could lift a representation over $\mathbb{F}_q$ to a representation over $\mathbb{C}$. Since representation is uniquely determined by its character up to isomorphism, this gives a map between representation rings $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$. Now take $G = GL_n(\mathbb{F}_q)$, consider the Brauer character χ_n of the standard representation of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. This gives a map $GL_n(\mathbb{F}_q) \to GL_n(\mathbb{C})$ hence a map $BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$. Compatibility ($\chi_n|_{GL_{n-1}} = \chi_{n-1}$) and universality of $BGL(\mathbb{F}_q)^+$ gives a map $BGL\mathbb{F}_q^+ \to BU$.

Then we show that this map composed with $\psi^q - 1$ is nullhomotopic hence induces a map to its homotopy fibre $F\psi^q$, call this map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. Both of these spaces are H-spaces and to show that θ is a homotopy equivalence, we show that it induces isomorphisms in homology groups. (This is referred to as Whitehead's thoerem for H-spaces. Another way to see this is to note that a homology isomorphism between H-spaces $f : X \to Y$ exhibits Y as a +-construction of X relative to the trivial subgroup of $\pi_1(X)$ and by universality of relative + construction [IV, Theorem 1.5, The K-Book, Weibel].)

To show that the map θ induces isomorphisms in integral homology, it suffices to show that it induces isomorphism in rational, mod p, and mod l homology for prime $(l, p) = 1$. The rational and mod p homology rings are both trivial [Section 3, Mitchell].

2 Rational and mod p homology

The cases for rational and mod p homology are easier to show. The reason that we need to separate the prime p and l is that in the mod p case, we use an argument of transfer in mod p homology but this does not work if we replace p with other prime, see lemma 2.1.

We have the following results.

Theorem 2.1. *(1)* $\tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Q}) = 0 = \tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Z}/p)$*;*
(2) $\tilde{H}_*(F\psi^q; \mathbb{Q}) = 0 = \tilde{H}_*(F\psi^q; \mathbb{Z}/p)$*.*

Let's first review some ideas of transfer in group cohomology.

Let G be a discrete group, M a $\mathbb{Z}G$-module and H a subgroup of G with finite index n. Let $g_1, g_2, \dots g_n$ be a set of left coset representatives of H in G. Then we have a natural transformation

$$\tau : M^H \to M^G$$

given by $\tau(m) = \Sigma g_i m$. It is obvious that τ is independent of the choice of coset representatives. To extend this definition to the derived functors, we simply apply it to the terms of a resolution. Explicitly, τ is the map induced on cohomology by the map of cochain complexes $\tau : (I_\bullet)^H \to (I_\bullet)^G$.

Proposition 2.1. *The composite of the maps* $H^*(G, M) \xrightarrow{i^*} H^*(H, M) \xrightarrow{\tau^*} H^*(G, M)$ *is multiplication by the index* $[G : H]$ *of* H *in* G*, where* i^* *and* τ^* *are the maps induced by inclusion and transfer respectively.*

A consequence of this is that if G is finite, and M is an RG-module, where $|G|$ is a unit in R. Then $H_n(G, M) = 0$ for all $n > 0$. From here, we have two consequences:

(a) For any finite group G, $\tilde{H}^*(G; \mathbb{Q}) = 0$. (b) If G is finite and $p \nmid |G|$, then $\tilde{H}^*(G; \mathbb{Z}/p) = 0$. The above proposition can be generalized.

Corollary 2.1. *Suppose* H *has finite index* d *in* G *and* M *is* $\mathbb{Z}[1/d]G$*-module. Then* $i^* : H^*(G, M) \to H^*(H, M)$ *is split injective.*

Proof. The splitting map is $1/d \cdot \tau$. □

A consequence of this corollary is:

(c) Suppose G is finite and H a subgroup which contains a p-Sylow subgroup of G. Then $i^* : H^*(G, \mathbb{Z}/p) \to H^*(H; \mathbb{Z}/p)$ is injective.

Proof of Theorem 2.1 (1) To show that $\tilde{H}_*(BGL\mathbb{F}_q^+;\mathbb{Q}) = 0$, it's enough to show that $\tilde{H}_*(BGL_n\mathbb{F}_q;\mathbb{Q}) = 0$ since homology commutes with direct limits. This follows from (a) above.

To show $\tilde{H}_*(BGL\mathbb{F}_q^+;\mathbb{Z}/p) = 0$, we need the following lemma.

Lemma 2.1. *$\tilde{H}_i(BGL_n\mathbb{F}_q^+;\mathbb{Z}/p) = 0$ for $i < d(p-1)$ and all n.*

Assuming this lemma, choose r prime to p, consider

$$BGL_n\mathbb{F}_q^+ \xrightarrow{i} BGL_n\mathbb{F}_{q^r}^+ \xrightarrow{\tau} BGL_n\mathbb{F}_q^+$$

where τ is the transfer map in algebraic K-theory. Let $(-)_p$ denote the localization at p. Then $(\tau i)_p$ is multiplication by r, hence is an equivalence since r is prime to p. Thus $H_*(\tau i;\mathbb{Z}/p)$ is an isomorphism. But applying the above lemma to the middle term $BGL_n\mathbb{F}_{q^r}^+$ of the above map shows that $H_n(\tau i, \mathbb{Z}/p)$ is the zero map for $n < dr(p-1)$. So $H_n(BGL\mathbb{F}_q;\mathbb{Z}/p) = 0$ for $n < dr(p-1)$, and since r was arbitrary this completes the proof.

Sketch proof of Lemma 2.1. The idea is that we want to consider some subgroups of $GL_n\mathbb{F}_q$ and use these subgroups to understand the homology groups of $BGL_n\mathbb{F}_q$. Note that the notation $H^*(BG;M)$ is the group cohomology $H^*(G;M)$ for a $\mathbb{Z}G$-module M. Let $B_n \subset GL_n$ be the subgroup of upper triangular matrices and let H_n be the subgroup of B_n consisting of all matrices whose diagonal entries are all 1. We show that B_n contains a p-Sylow subgroup H_n of $GL_n\mathbb{F}_q$ thus by an argument of transfer in cohomology [see consequence (c) below corollary 2.1], we have that the restriction map $H^*(BGL_n\mathbb{F}_q;\mathbb{Z}/p) \to H^*(B_n;\mathbb{Z}/p)$ is injective. The lemma is proved by showing that

$$\tilde{H}_i(B_n) = 0 \text{ for } i < d(p-1).$$

To show this, we proceed by induction. The case $n = 1$ is $B_1 = \mathbb{F}_q^\times$, $\tilde{H}_i B_1 = 0$ for all i since $p \nmid |B_1|$. At the inductive step, consider the group extension

$$A_n \to B_n \to B_{n-1}$$

where A_n is the "top row" subgroup. If we can show that

$$\tilde{H}_i(A_n) = 0 \text{ for } i < d(p-1),$$

then we can use the Hochschild-Serre spectral sequence to finish the inductive step. To show the result for A_n, note that A_n is a semidirect product of the form $V \to A_n \to \mathbb{F}_q^\times$ where V is the additive group of a vector space over $\mathbb{F}_q$ and $\mathbb{F}_q^\times$ acts on V by scalar multiplication. We use the cohomology spectral sequence of this extension to compute the cohomology groups of A_n. □

3 Mod l homology

In this section, we outline the proof of the following theorem:

Theorem 3.1. *$\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ induces an isomorphism on $H_*(-;\mathbb{Z}/l)$.*

Let r be minimal such that $l|q^r-1$. Thus r is the order of q in the group $\mathbb{Z}/l^{\times}$, so $r|l-1$. Let a be maximal such that $l^a|q^r-1$. Let μ be the group of l^a-th roots of unity which is the l-torsion subgroup of $\mathbb{F}_{q^r}^{\times}$. Now let C_q denote the subgroup of $GL_r\mathbb{F}_q$ generated by μ and the Galois group $G(\mathbb{F}_{q^r}/\mathbb{F}_q)$. Here we have identified $(\mathbb{F}_q)^r$ with $\mathbb{F}_{q^r}$. The Galois group is cyclic of order r, generated by the Frobenius σ. Thus C_q fits into a split extension

$$\mu \xrightarrow{i} C_q \to \mathbb{Z}/r$$

where σ acts on μ by $\sigma(\alpha)=\alpha^q$.

Moreover, we can first reduce to the case $l|q-1$ [See lemma 4.3 ,[1]], in this case $B\mu = BC_q$.

For the rest of this section, we assume $l|q-1$.

Now the inclusion $C_q \subset GL_r\mathbb{F}_q$ induces a map $BC_q \to BGL\mathbb{F}_q^+$ and hence $H_*BC_q \to H_*BGL\mathbb{F}_q^+$. Since $BGL\mathbb{F}_q^+$ is a homotopy associative and commutative H-space, this map in turn extends to a ring homomorphism $S(\tilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$, where $S(-)$ denotes the symmetric algebra. Let $S'(-)$ denote the strict symmetric algebra, that is, the quotient of the symmetric algebra obtained by factoring out the ideal generated by all a^2 with $|a|$ odd. This refinement is only relevant when $l=2$, since for l odd $S=S'$.

3.1 Homology ring of $F\psi^q$

We want to show the isomorphism in mod l homology by starting with calculating the mod l homology ring of $F\psi^q$ in the case $l|q-1$. The good thing of the case $l|q-1$ is that $B\mu = BC_q$ and μ is a cyclic group.

Note that we know the homology ring of U and BU:

$$H_*BU \cong \mathbb{Z}/l[c_1, c_2, \dots]$$

where $|c_i| = 2i$ and c_i is the i-th Chern class; and

$$H_*U \cong \mathbb{Z}/l\langle c_1, c_2, \dots\rangle$$

with $|x_i| = 2i-1$.

The following proposition comes from Lemma 4.4, 4.5,[1]. I rewrote the proofs adding details of "delooping" to 4.4 and shortening the proof to 4.5.

Main point: From the above results of homology rings of U and BU, it's not hard to see that the spectral sequence associated to the fibre sequence $U \to F\psi^q \to BU$ collapses at E_2. To show that the maps in the spectral sequence are homomorphisms of algebras, we want to show that the fibre sequence can be double-delooped since the multiplication structure of the H-spaces are induced by the double loop structure. The Adams operation ψ^q does not commute with the Bott map but it does when we localize at l for l coprime to p.

Proposition 3.1. *If $l|q-1$, then $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras.*

Proof. Consider the fibre sequence $U \to F\psi^q \to BU$ coming from the fibre sequence $F\psi^q \to BU \to BU$. Let $L : X \mapsto X_{(l)}$ denote the localization of X away from l. Let $\beta : BU \to \Omega_0^2 BU$ denote the Bott map, then we form a diagram where $h = L(\psi^q - 1)$

$$\begin{array}{ccc} BU & \xrightarrow{\psi^q - 1} & BU \\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle L} \\ BU_{(l)} & \xrightarrow{h} & BU_{(l)} \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta} \\ \Omega_0^2 BU_{(l)} & \xrightarrow{\Omega_0^2 h} & \Omega_0^2 BU_{(l)} \end{array}$$

Note that the Bott isomorphism $\beta : \tilde{K}X \xrightarrow{\cong} \tilde{K}(S^2 \wedge X)$ does not commute with the Adams operations. In fact, we have the formula $\psi^q(\beta a) = q\beta(\psi^q a)$ because $\beta a = b \times a$ with $b \in \tilde{K}S^2$ and ψ^q is multiplicative. Going around the right side gives $a \mapsto \beta\psi^q a - \beta a = 1/q\psi^q\beta a - \beta a$ and going around the left side gives $a \mapsto \psi^q\beta a - \beta a$. Moreover, multiplication by q is an equivalence for $BU_{(l)}$ since $(l, q) = 1$. We have that the above diagram homotopy commutes and h is equivalent to $\Omega_0^2 h$, a double-loop map. Now the fibre sequence associated to h is multiplicative meaning that h commutes with the H-space multiplication, thus the spectral sequence is a spectral sequence of Hopf algebras. Since L induces isomorphisms in mod l homologies, the same is true for $\psi^q - 1 : BU \to BU$.

Consider the fibre sequence mod l homology associated to $F\psi^q \to BU \to BU$. Since $\pi_1 BU$ is trivial, the E_2 term of this fibre sequence is $H_*U \otimes H_*BU$. We know that $H_*BU \cong \mathbb{Z}/l[c_1, c_2, \dots]$ where $|c_i| = 2i$ the Chern classes and $H_*U \cong \mathbb{Z}/l\langle x_1, x_2, \dots\rangle$ with $|x_i| = 2i-1$. Since the bidegree $(r, -r+1)$ of a differential map is odd for one and even for the other, by observing the degrees of nonzero $H_*U \otimes H_*BU$, the spectral sequence collapses at E_2. Thus the mod l homology $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras. □

3.2 Homology ring of $BGL\mathbb{F}_q^+$

Now we know the homology ring of $F\psi^q$. Recall at the beginning of this section the inclusion $C_q \subset GL_r\mathbb{F}_q$ induces a map $BC_q \to BGL\mathbb{F}_q^+$ and hence $H_*BC_q \to H_*BGL\mathbb{F}_q^+$. And in the case $l|q-1$ we have $B\mu \cong BC_q$. Since μ is cyclic, we can compute its group cohomology to be $H^*\mu = \mathbb{Z}/l[y] \otimes \mathbb{Z}/l\langle x\rangle$ with $|y| = 2$ and $|x| = 1$. Compared with the homology ring for $F\psi^q$, naturally we want to show that $B\mu$ is a generating complex.

Theorem 3.1. *The natural map $S(\tilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$ induces an isomorphism $S'(\tilde{H}_*BC_q) \xrightarrow{\cong} H_*BGL\mathbb{F}_q^+$, here $S(V)$ denote the symmetric algebra of a graded vector space V and $S'(V)$ denote the strict symmetric algebra of V.*

Theorem 3.2 (Theorem 4.6, [1])**.** *Let j be the composite $B\mu \to BGL\mathbb{F}_q^+ \to F\psi^q$. This map induces an isomorphism $S'(\tilde{H}_*B\mu) \xrightarrow{\cong} H_*F\psi^q$.*

In the case l odd, symmetric and strict symmetric algebra are the same and $BC_q = B\mu$ when $l|q-1$ (we've reduced to the case $l|q-1$), so from the following

$$S(\tilde{H}_*B\mu) \xrightarrow{\varphi} H_*BGL\mathbb{F}_q^+ \xrightarrow{\theta_*} H_*F\psi^q,$$

we have that $\theta_* : H_*BGL_n\mathbb{F}_q^+ \to H_*F\psi^q$ is an isomorphism since φ and the composite $\theta_*\varphi$ are both isomorphism from Theorem 3.2.

The proofs of Theorem 3.1 and 3.2 require much more work and we refer to [1], [2] for the details. In the case $l = 2$, we need to show that $e_i = 0$ in $H_*BGL_n\mathbb{F}_q^+$, where e_i are generators of $H_{2i-1}B\mu$. This requires another ingeneous work to complete, see [Section 7, [1]] or [2].

We'd also like to mention that the calculations for homology rings are directly aimed at the computations of K-groups. If we carry out these calculations further, we can derive some further computations on the homology and cohomology of $GL_n\mathbb{F}_q$, see [2].

References

[1] Mitchell, S. Notes on K theory of finite fields. Available online: https://sites.math.northwestern.edu/ jnkf/Mitchell-finitefieldsKtheory.pdf

[2] Quillen, D. (1972). On the cohomology and K-theory of the general linear groups over a finite field. *Annals of Mathematics*, 96(3), 552-586.