On the Lichtenbaum-Quillen Conjectures

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Abstract

Starting with some motivations and brief expositions on algebraic K-theory, I'll introduce some early important computations of algebraic K-theory, including computations of K-theory of finite fields and of rings of integers for which I will briefly outline the proofs. Then we'll move on to K-theory with finite coefficients of separably closed fields. With the motivation of recovering some information of K-theory of an arbitrary field from its separable closure, we introduce a few versions of the Lichtenbaum-Quillen conjectures as descent spectral sequences of étale Cohomology groups. If time permits, I'll mention relation to motivic Cohomology that a key tool is some "motivic-to-K-theory" spectral sequence.

These are notes based on my talk on Oct 01, 2021 in UIUC Graduate Homotopy Seminar. The main references are [1] and [2]. The outline of the proof of Quillen's K-theory of finite fields has been moved to Appendix A.

Contents

1 Lower K-Theory

Algebraic K-theory originated from constructing invariants for rings. To study a ring, we want to understand modules over the ring. Only in a few cases, we can actually classify all modules over a ring, say in the case of principal ideal domain. Naturally, we want to construct an invariant for some category of modules over a ring. The category of all modules over a ring is a monoid with monoid product given by direct sum. But this is not a group without the desired property of cancellation. To form a nice object of study, we want to construct a group generated from a monoid.

Definition 1.1. *The* ***group completion*** *of a monoid M is an abelian group $M^{-1}M$ together with a monoid map $M \to M^{-1}M$ which is universal in the sense that for any abelian group A ad any monoid map $\alpha : M \to A$, there exists a unique group homomorphism $\tilde{\alpha} : M^{-1}M \to A$ such that the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{[-]} & M^{-1}M \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \exists ! \tilde{\alpha}} & \\ A & & \end{array} .$$

One way to construct the group completion of a monoid M is to form the free abelian group $F(M)$ on symbols $[m], m \in M$, and then factor out by the subgroup $R(M)$ generated by the relations $[m+n] - [m] - [n]$.

The idea of group completion can be applied to any category with a natural product making the isomorphism classes of objects into an abelian monoid.

Let S be a category, suppose the isomorphism classes of objects of S forms a set, denoted by S^{iso}. If S is symmetric monoidal, the set S^{iso} is an abelian monoid.

Definition 1.2. *Let S be a symmetric monoidal category and S^{iso} be the abelian monoid of isomorphism classes of S. The group completion of S^{iso} is called the* ***Grothendieck group*** *of S, denoted as $K_0(S)$.*

Let R be an associative ring with unit 1. Define MR to be the category of finitely generated modules over the ring R, and PR to be the category of finitely generated projective modules over the ring R.

Define $K_0(R) := K_0PR$ and $G_0(R) := K_0MR$.

Why don't we take K_0 of the category of all modules? Because the group completion will be trivial due to $[M] + [M^\infty] = [M^\infty]$ for any module M. The same thing happens for the category of all projective modules too: for any projective module P, let Q be a module such that $P \oplus Q = R^n$ is free (n can be ∞), then [Eilenburg-Swindle]

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = R^\infty.$$

Naturally we want to restrict to the category of finitely generated modules. Since projective modules are very nice algebraic objects to study, $K_0(R)$ is incredibly interesting to study. Also, $K_0(R)$ and $M_0(R)$ are closely related with each other since every module has a projective resolution.

Example 1.1.

1. Let R be a field, a division ring or a principal ideal domain, then $K_0(R) \cong G_0(R) \cong \mathbb{Z}$ generated by rank 1 free module.

2. (Serre) If R is the coordinate ring of an affine algebraic variety V, then $PR \cong \mathrm{Vect}V$ where $\mathrm{Vect}V$ is the category of algebraic vector bundles over V (finite rank locally free sheaves), $K_0(R) \cong K_0(\mathrm{Vect}V)$.

3. (Swan) Let R be the ring of continuous functions on a compact Hausdorff space X, $PR \cong \mathrm{Vect}X$ where $\mathrm{Vect}V$ is the category of vector bundles over X. Hence, $K_0(R)$ coincides with topological K-theory $K(X)$.

4. In the case that $K_0(R)$ and $G_0(R)$ are isomorphic, the canonical map induced by inclusion need not be an isomorphism. For example, let $R = \mathbb{Z}/p^n$, both groups are $\mathbb{Z}$, $K_0(R)$ is generated by the rank 1 free module $[\mathbb{Z}/p^n]$ and $G_0(R)$ is generated by the length 1 module $[\mathbb{Z}/p]$, the map $K_0(R) \to G_0(R)$ takes the generator $[\mathbb{Z}/p^n]$ to $n[\mathbb{Z}/p]$.

In the case of abelian categories, we will be able to use some general machineries to study the K-groups. First we define K_0 for an abelian category.

Definition 1.3. *Let $\mathcal{A}$ be an abelian category, define the* ***Grothendieck group of*** $\mathcal{A}$ *to be the abelian group presented as having one generator $[A]$, $\forall A \in \mathcal{A}$ with the relation $[A] = [A'] + [A'']$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$.*

Theorem 1.1 (Localization, II.6.4, [2])**.** *Let $\mathcal{B}$ be a Serre subcategory of a (small) abelian category $\mathcal{A}$. Then the following sequence is exact:*

$$K_0(B) \to K_0(A) \to K_0(A/B) \to 0.$$

This sequence can indeed be extended to higher K-groups. Note that when we study lower K-groups, the K-groups can be defined for different types of categories which can possibly give different short exact sequences in different sense. Historically, one particular exact sequence that was aimed at extending was the one related to the ring of integers.

1.1 K_1 of a ring

Let R be an associative ring with unit. Identifying each $n \times n$ matrix g with the larger matrix $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ gives an embedding of $GLn(R)$ into $GL_{n+1}(R)$. The union of the resulting sequence $GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \dots$ is called the infinite general linear group $GL(R)$. Recall that the commutator subgroup $[G, G]$ of a group G is the subgroup generated by its commutators $[g, h] = ghg^{-1}h^{-1}$. It is always a normal subgroup of G and has a universal property: the quotient $G/[G, G]$ is an abelian group, and every homomorphism from G to an abelian group factors through $G/[G, G]$.

Definition 1.4. *$K_1(R)$ is the abelian group $GL(R)/[GL(R), GL(R)]$.*

The universal property of $K_1(R)$ is this: every homomorphism from $GL(R)$ to an abelian group must factor through the natural quotient $GL(R) \to K1(R)$. Note that $K_1(R)$ is $H_1(BGLR; \mathbb{Z})$ since $BGLR$ is a $K(GLR, 1)$ space.

1.2 K_2 of a ring

Now we define a group in terms of generators and relations designed to imitate the behaviour of the elementary matrices). To avoid technical complications, we shall avoid any definition of $St_2(R)$.

Definition 1.5. *For $n \geq 3$ the Steinberg group $St_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$ with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations":*

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$

$$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell, \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell, \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\varphi_n : St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$. The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\lim\limits_{\to} St_n(R)$ and observe that by stabilizing, the ϕ_n induce a surjection $\phi : St(R) \to E(R)$.

Definition 1.6. *The group $K_2(R)$ is defined by*

$$K_2(R) := \ker(\phi : St(R) \to E(R)).$$

Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

Theorem 1.2 (Kervaire, Steinberg, Theorem III. 5.5, [2])**.** *The Steinberg group $St(R)$ is the universal central extension of $E(R)$. Hence $K_2(R) \cong H_2(E(R);\mathbb{Z})$.*

1.3 Exact sequences relating K_0, K_1 and K_2

Let F be a number field, $S \subset \mathrm{Spec}\mathcal{O}_F \setminus \{0\}$. Then there exists an exact sequence

$$\bigoplus_{\mathcal{P}\in S} K_2\mathcal{O}_F/\mathcal{P} \to K_2\mathcal{O}_F \to K_2S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P}\in S} K_1\mathcal{O}_F/\mathcal{P} \to K_1\mathcal{O}_F \to$$

$$K_1S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P}\in S} K_0\mathcal{O}_F/\mathcal{P} \to K_0\mathcal{O}_F \to K_0S^{-1}\mathcal{O}_F \to 0.$$

Note the special cases:

1. $S = \mathrm{spec}\,\mathcal{O}_F \setminus \{0\}$, $S^{-1}\mathcal{O}_F = F$.
2. $S^{-1}\mathcal{O}_F = \mathcal{O}_F[1/l]$, $S =$ the set of primes over l.

One usefulness of this exact sequence is Bass-Milnor-Serre theorem which was proved by showing that $K_2F \to \underset{p\in S}{\oplus} K_1\mathcal{O}_F/p$ is onto [[3]]. One motivation is to extend this exact sequence to the left which is indeed compatible with the extension of the exact sequence for K_0 of abelian categories in Theorem 1.1.

2 Higher K-theory

Quillen proposed several equivalent definitions of the higher K-groups of rings, symmetric monoidal categories, abelian categories: the plus construction, group completion and the Q-construction. Two points to note:

1. The K-group in every case is defined to be the homotopy groups of a space;
2. The space is indeed an infinite loop space, so the space $K_*(R)$ is indeed the homotopy groups of a spectrum.

Definition 2.1 (BGL^+ for rings)**.** *The notation $BGL(R)^+$ will denote any CW complex X which has a distinguished map $BGL(R) \to BGL(R)^+$ such that the following are true:*

(1) $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$, (i.e. the map $BGLR \to BGLR^+$ is the abelianization on π_1);

(2) $H^*(BGL(R); M) \xrightarrow{\cong} H^*(BGL(R)^+; M)$ for every $K_1(R)$-module M.

For $n \geq 1$, $K_n(R)$ is defined to be the homotopy group $\pi_n BGL(R)^+$.

Two main calculations by Quillen:

1. [Quillen 1972, [4]] For every finite fields $\mathbb{F}_q$ and $n \geq 1$, we have

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i - 1 \\ 0, & n \text{ even.} \end{cases} \tag{1}$$

2. [Quillen 1973, [5]] Let F be a number field, $S \subset \mathrm{Spec}\mathcal{O}_F \setminus \{0\}$. Then there exists an exact sequence

$$\ldots \to \bigoplus_{\mathcal{P} \in S} K_n(\mathcal{O}_F/\mathcal{P}) \to K_n\mathcal{O}_F \to K_n S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P} \in S} K_{n-1}\mathcal{O}_F/\mathcal{P} \to \ldots$$

For a summary of computation of K-theory of finite fields, see Appendix A. The second computation follows from a more general results for abelian categories:

Theorem 2.1 (Theorem V.5.1, [2])**.** *Let $\mathcal{B}$ be a Serre subcategory of a (small) abelian category $\mathcal{A}$. Then $K(\mathcal{B}) \to K(\mathcal{A}) \xrightarrow{loc} K(\mathcal{A}/\mathcal{B})$ is a homotopy fibration sequence. Thus there is a long exact sequence of homotopy groups*

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n(\mathcal{B}) \to K_n(\mathcal{A}) \xrightarrow{loc} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{n-1}(\mathcal{B}) \to \ldots$$

ending in $K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$, the exact sequence in Theorem 1.1.

In general, PR is not an abelian category, for example, it may not have cokernel. While MR become an abelian category when R is noetherian, thus the above exact sequence applies to MR in this case. In the case that R is regular ($\mathcal{O}_F$ is regular), every R-module has a finite projective resolution. In this case, we can apply Resolution Theorem [V.3.1, [2]] to identify $K_n(PR)$ and $K_n(MR)$.

Moreover, let $\phi : F \to E$ be a finite field extension of degree d. Fixing a basis for the extension determines a map $GL_n E \to GL_{dn} F$ which induces *the transfer* map for K-theory $\tau : BGLE^+ \to BGLF^+$. Let $i : BGLF^+ \to BGLE^+$ be the map induced from inclusion, then the map $\pi_*(\tau i)$ is multiplication by d. Suppose E/F is Galois with Galois group $G = Gal(R/F)$, then the composite $BGLE^+ \xrightarrow{\tau} BGLF^+ \xrightarrow{i} BGLE^+$ is homotopic to $\Sigma_{g \in G} g$. Moreover, for a fixed prime l with $l \nmid d$, by localizing at l, we can show that $K_*F = (K_*E)^G$ is the "fixed point" spectrum of G by which we mean the mapping telescope of the idempotent $1/d\Sigma_{g \in G} g$ [P.181, 2.13, [1]].

Up to now, the only fields whose K-theory we can compute are the algebraic extension of finite fields. What about algebraically closed fields?

Quillen and Lichtenbaum conjectured that if F is algebraically closed and char $F \neq l$, $K_*/l^\nu F$ should be the same as ordinary topological K-theory.

Ten years later, the conjectures were proved by Suslin in two papers [[6], [7]]:

Theorem 2.2 ([6])**.** *Let $i : F \to E$ be an extension of algebraically closed fields. Then $i_* : K_*(F; \mathbb{Z}/l) \xrightarrow{\cong} K_*(E; \mathbb{Z}/l)$ is an isomorphism for all n.*

By this theorem, it's enough to compute $K_*(F;\mathbb{Z}/n)$ for one algebraically closed fields of each characteristic. By Quillen's work, this settles the case char $F = p$. It remains to compute $K_*(F;\mathbb{Z}/n)$ for some F of characteristic 0.

Before the second theorem of Suslin, we begin with some general remarks. If G is a topological group, then it is important to distinguish between the classifying space BG^δ of the discrete group G, and the classifying space BG^{top} of the topological group G^{top} [see IV.3.9, [2]] . For example, the homotopy groups of $BGL(C)^\delta$ are zero, except for the fundamental group, while the homotopy groups of $BGL(C)^{top} \cong BU$ are given by Bott periodicity.

Theorem 2.3 ([7])**.** *The natural map*

$$K_*(\mathbb{C};\mathbb{Z}/n) \to K_*(\mathbb{C}^{top};\mathbb{Z}/n)$$

is an isomorphism for all n.

Moreover, given a separably closed field F of characteristic p, then $K_n(F)$ is noncanonically a summand of $K_n(\bar{F})$ by a transfer argument. Hence we could recover some structures of F from its algebraic closure, see the Remark below Thoerem VI.1.3, [2].

2.1 The Lichtenbaumn-Quillen Conjectures

Given an arbitrary field F, we hope to somehow recover $K_*(F;\mathbb{Z}/l^\nu)$ from $K_*(\bar{F};\mathbb{Z}/l^\nu)$ where $\bar{F}$ is its separable closure, see IV.2, [2] for the definition of K-theory with finite coefficients.

If E/F is any Galois extension, $BGLF^+$ is exactly the fixed point set of $G = Gal(E/F)$, $BGLF^+ = (BGLE^+)^G$. A naive guess is that we may have $K_*(F;l^v) = (K_*(E;l^\nu))^G$. This is not true indeed. A more reasonable hope would be that there is a descent spectral sequence

$$E_2^{p,q} = H_p(GL(E/F), K_q(E;l^\nu)) \Rightarrow K_{q-p}(F;l^\nu).$$

Whether such spectral sequence exists had played a central role in homotopy theory for decades.

We now turn to the general descent question for a scheme X. Write $1/l \in X$ as shorthand for "the residue field characteristic of X are all prime to l". Here is the first version of Lichtenbaum-Quillen conjectures. Here, sufficiently nice should include at least the following: (a) $1/l \in X$; (b) X is regular; $cd_l X < \infty$.

Theorem 2.4 (LQCI, 5.12, [1])**.** *If X is a sufficiently nice scheme, then there is a descent spectral sequence with*

$$\begin{aligned} E_2^{p,q} &= H^p_{\acute{e}t}(X, \mu_{l^\nu}(i))(q = 2i) \\ &= 0(q \ odd) \end{aligned}$$

converging to $K_{p-q}(X;l^\mu)$ if $q - p$ is sufficiently large. Here $p - q \geq 1$ should suffice for $X = \operatorname{Spec} \mathcal{O}_F[1/l]$.

Remark. *Even for $X = \operatorname{Spec} F$, there is no such spectral sequence converging precisely to $K_*(X, l^\mu)$. If taking X to be a smooth projective variety over $\mathbb{C}$, the descent spectral sequence would have to be the Atiyah-Hirzebruch spectral sequence for ordinary topological K-theory mod l^μ. Hence if it converges strictly then topological and algebraic K-theory would be the same but this is not the case.*

Another version of LQC involves an auxiliary space that does have descent which is called the l-adic étale K-theory spectrum $K^{\acute{e}t}X$ and the étale K-theory is defined as $K^{\acute{e}t}_n X = \pi_n K^{\acute{e}t} X$. It has the following properties:

Suppose $\mathrm{cd}_l X < \infty$. Then there is a strongly convergent spectral sequence

$$E_2^{p,q} = \left\{ \begin{matrix} H^p_{\text{ét}}(X, \mathbb{Z}/\ell^v(i)) & (q = 2i) \\ 0 & (q \text{ odd}) \end{matrix} \right\} \Longrightarrow K^{\text{ét}}_{q-p}(X; \mathbb{Z}/\ell^v)$$

Here sufficiently nice has the same vague meaning as in LQCI except that we don't assume $cd_l X < \infty$. Below is a reformulation of LQC:

Theorem 2.5. *(LQCII, 7.3, [1]) If X is a sufficiently nice scheme, there is a map $\varphi : KX \to K^{\acute{e}t}X$ induces an isomorphism $K_n(X;\mathbb{Z}/l) \to K^{\acute{e}t}_n(X;\mathbb{Z}/l)$ for all n sufficiently large. Here $n \geq 1$ should suffice for $X = \operatorname{spec} \mathcal{O}_F[1/l]$.*

For a proof of the Lichtenbaumn-Quillen conjecture after motivic homotopy theory, see [8].

2.2 The Motivic-to-K-theory Spectral Sequence

In 1990's Grayson constructed a spectral sequence converging to K-theory of regular rings [[9]], it was not clear what the E_2 terms were.

In 2001 Suslin showed (in [[10]]) that the E_2 terms in Grayson's spectral sequence agreed with motivic cohomology for fields.

In 2000-2001, Voevodsky obeserved the existence of a spectral sequence of the following form converging to K-theory modulo two conjectures in motivic homotopy theory since verified [[11], [12]]. A third constructions was given by Levine [[13], [14]] who also proved these three spectral sequences agree in [14].

For every abelian group A and every (n, i), the motivic cohomology of a smooth scheme X over a field consists of groups $H^n(X, A(i))$, defined as a hypercohomology of a certain chain complex $A(i)$ of Nisnevich sheaves, see [[15]] for the definitions and properties of motivic cohomology.

Theorem 2.6 (VI.4.2, [2]). *For any coefficient group A and any smooth scheme X over a field k, there is a spectral sequence, natural in X and A:*

$$E_2^{p,q} = H^{p-q}(X, A(-q)) \Rightarrow K_{-p-q}(X; A).$$

If $X = \operatorname{spec} k$ and $A = \mathbb{Z}/m$ where $1/m \in k$, then the E_2 terms are just the étale cohomology groups of k, truncated to lie in the octant $q \leq p \leq 0$.

All the above computations can be quickly recovered from the degeneracy of this spectral sequence at the E_2 page, see VI, Examples 4.5, [2] for such calculations.

A K-theory of Finite Fields

This is an outline of Quillen's proof for the calculation of K-theory of finite fields, originally done by Quillen in [17], see also [16] for a slightly different presentation with more background materials included.

Let $\mathbb{F}_q$ denote the field with q elements, where $q = p^d$, p a prime.

A.1 Back story

Here is Quillen's theorem on K theory of finite fields. Below I'll summarize and outline the main points of the proof.

Theorem.

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i - 1 \\ 0, & n \text{ even.} \end{cases} \tag{2}$$

Let $\psi^q : BU \to BU$ be the map realizing Adams operation on $\tilde{K}X$ and $\chi : BU \to BU$ be the map realizing the inverse of BU (BU is an H-space and let m denote the multiplication).

We form the difference $\psi^q - 1$:

$$BU \xrightarrow{\Delta} BU \times BU \xrightarrow{\psi^q \times \chi} BU \times BU \xrightarrow{m} BU$$

and let $F\psi^q$ be the homotopy fibre of this map.

The homotopy groups of $F\psi^q$ are the same as in (1). This follows from the long exact sequence of the fibre sequence $F\psi^q \to BU \xrightarrow{\psi^q - 1} BU$ and that the Adams operation ψ^k on $\widetilde{KU}(S^{2n})$ is multiplication by k^n.

Next we construct a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ [Section 3, Mitchell] as follow. By Green's theoerm, for any finite group G and any representation of G over $\mathbb{F}_q$, the Brauer character is a virtual complex character, that is, the character of a virtual complex representation. This means that we could lift a representation over $\mathbb{F}_q$ to a representation over $\mathbb{C}$. Since representation is uniquely determined by its character up to isomorphism, this gives a map between representation rings $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$. Now take $G = GL_n(\mathbb{F}_q)$, consider the Brauer character χ_n of the standard representation of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. This gives a map $GL_n(\mathbb{F}_q) \to GL_n(\mathbb{C})$ hence a map $BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$. Compatibility ($\chi_n|_{GL_{n-1}} = \chi_{n-1}$) and universality of $BGL(\mathbb{F}_q)^+$ gives a map $BGL\mathbb{F}_q^+ \to BU$.

Then we show that this map composed with $\psi^q - 1$ is nullhomotopic hence induces a map to its homotopy fibre $F\psi^q$, call this map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. Both of these spaces are H-spaces and to show that θ is a homotopy equivalence, we show that it induces isomorphisms in homology groups. (This is referred to as Whitehead's thoerem for H-spaces. Another way to see this is to note that a homology isomorphism between H-spaces $f : X \to Y$ exhibits Y as a +-construction of X relative to the trivial subgroup of $\pi_1(X)$ and by universality of relative + construction [IV, Theorem 1.5, The K-Book, Weibel].)

To show that the map θ induces isomorphisms in integral homology, it suffices to show that it induces isomorphism in rational, mod p, and mod l homology for prime $(l, p) = 1$. The rational and mod p homology rings are both trivial [Section 3, Mitchell].

A.2 Rational and mod p homology

The cases for rational and mod p homology are easier to show. The reason that we need to separate the prime p and l is that in the mod p case, we use an argument of transfer in mod p homology but this does not work if we replace p with other prime, see lemma A.1.

We have the following results.

Theorem A.1. *(1)* $\tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Q}) = 0 = \tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Z}/p)$*;*

(2) $\tilde{H}_*(F\psi^q; \mathbb{Q}) = 0 = \tilde{H}_*(F\psi^q; \mathbb{Z}/p)$.

Let's first review some ideas of transfer in group cohomology.

Let G be a discrete group, M a $\mathbb{Z}G$-module and H a subgroup of G with finite index n. Let $g_1, g_2, \ldots g_n$ be a set of left coset representatives of H in G. Then we have a natural transformation

$$\tau : M^H \to M^G$$

given by $\tau(m) = \Sigma g_i m$. It is obvious that τ is independent of the choice of coset representatives. To extend this definition to the derived functors, we simply apply it to the terms of a resolution. Explicitly, τ is the map induced on cohomology by the map of cochain complexes $\tau : (I_\bullet)^H \to (I_\bullet)^G$.

Proposition A.1. *The composite of the maps* $H^*(G, M) \xrightarrow{i^*} H^*(H, M) \xrightarrow{\tau^*} H^*(G, M)$ *is multiplication by the index* $[G : H]$ *of* H *in* G*, where* i^* *and* τ^* *are the maps induced by inclusion and transfer respectively.*

A consequence of this is that if G is finite, and M is an RG-module, where $|G|$ is a unit in R. Then $H_n(G, M) = 0$ for all $n > 0$. From here, we have two consequences:

(a) For any finite group G, $\tilde{H}^*(G; \mathbb{Q}) = 0$. (b) If G is finite and $p \nmid |G|$, then $\tilde{H}^*(G; \mathbb{Z}/p) = 0$. The above proposition can be generalized.

Corollary A.1. *Suppose* H *has finite index* d *in* G *and* M *is* $\mathbb{Z}[1/d]G$*-module. Then* $i^* : H^*(G, M) \to H^*(H, M)$ *is split injective.*

Proof. The splitting map is $1/d \cdot \tau$. □

A consequence of this corollary is:

(c) Suppose G is finite and H a subgroup which contains a p-Sylow subgroup of G. Then $i^* : H^*(G, \mathbb{Z}/p) \to H^*(H; \mathbb{Z}/p)$ is injective.

Proof of Theorem A.1 (1) To show that $\tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Q}) = 0$, it's enough to show that $\tilde{H}_*(BGL_n\mathbb{F}_q; \mathbb{Q}) = 0$ since homology commutes with direct limits. This follows from (a) above.

To show $\tilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Z}/p) = 0$, we need the following lemma.

Lemma A.1. $\tilde{H}_i(BGL_n\mathbb{F}_q^+; \mathbb{Z}/p) = 0$ *for* $i < d(p-1)$ *and all* n*.*

Assuming this lemma, choose r prime to p, consider

$$BGL_n\mathbb{F}_q^+ \xrightarrow{i} BGL_n\mathbb{F}_{q^r}^+ \xrightarrow{\tau} BGL_n\mathbb{F}_q^+$$

where τ is the transfer map in algebraic K-theory. Let $(-)_p$ denote the localization at p. Then $(\tau i)_p$ is multiplication by r, hence is an equivalence since r is prime to p. Thus $H_*(\tau i; \mathbb{Z}/p)$ is an isomorphism. But applying the above lemma to the middle term $BGL_n\mathbb{F}_{q^r}^+$ of the above map shows that $H_n(\tau i, \mathbb{Z}/p)$ is the zero map for $n < dr(p-1)$. So $H_n(BGL\mathbb{F}_q; \mathbb{Z}/p) = 0$ for $n < dr(p-1)$, and since r was arbitrary this completes the proof.

Sketch proof of Lemma A.1. The idea is that we want to consider some subgroups of $GL_n\mathbb{F}_q$ and use these subgroups to understand the homology groups of $BGL_n\mathbb{F}_q$. Note that the notation $H^*(BG; M)$ is the group cohomology $H^*(G; M)$ for a $\mathbb{Z}G$-module M. Let $B_n \subset GL_n$ be the subgroup of upper triangular matrices and let H_n be the subgroup of B_n consisting of all matrices whose diagonal entries are all 1. We show that B_n contains a p-Sylow subgroup H_n of $GL_n\mathbb{F}_q$ thus by an argument of transfer in cohomology [see consequence (c) below corollary A.1], we have that the restriction map $H^*(BGL_n\mathbb{F}_q;\mathbb{Z}/p) \to H^*(B_n;\mathbb{Z}/p)$ is injective. The lemma is proved by showing that

$$\tilde{H}_i(B_n) = 0 \text{ for } i < d(p-1).$$

To show this, we proceed by induction. The case $n = 1$ is $B_1 = \mathbb{F}_q^\times$, $\tilde{H}_i B_1 = 0$ for all i since $p \nmid |B_1|$. At the inductive step, consider the group extension

$$A_n \to B_n \to B_{n-1}$$

where A_n is the "top row" subgroup. If we can show that

$$\tilde{H}_i(A_n) = 0 \text{ for } i < d(p-1),$$

then we can use the Hochschild-Serre spectral sequence to finish the inductive step. To show the result for A_n, note that A_n is a semidirect product of the form $V \to A_n \to \mathbb{F}_q^\times$ where V is the additive group of a vector space over $\mathbb{F}_q$ and $\mathbb{F}_q^\times$ acts on V by scalar multiplication. We use the cohomology spectral sequence of this extension to compute the cohomology groups of A_n. □

A.3 Mod l homology

In this section, we outline the proof of the following theorem:

Theorem A.1. *$\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ induces an isomorphism on $H_*(-;\mathbb{Z}/l)$.*

Let r be minimal such that $l|q^r - 1$. Thus r is the order of q in the group $\mathbb{Z}/l^\times$, so $r|l-1$. Let a be maximal such that $l^a|q^r - 1$. Let μ be the group of l^a-th roots of unity which is the l-torsion subgroup of $\mathbb{F}_{q^r}^\times$. Now let C_q denote the subgroup of $GL_r\mathbb{F}_q$ generated by μ and the Galois group $G(\mathbb{F}_{q^r}/\mathbb{F}_q)$. Here we have identified $(\mathbb{F}_q)^r$ with $\mathbb{F}_{q^r}$. The Galois group is cyclic of order r, generated by the Frobenius σ. Thus C_q fits into a split extension

$$\mu \xrightarrow{i} C_q \to \mathbb{Z}/r$$

where σ acts on μ by $\sigma(\alpha) = \alpha^q$.

Moreover, we can first reduce to the case $l|q-1$ [See lemma 4.3 ,[16]], in this case $B\mu = BC_q$.

For the rest of this section, we assume $l|q-1$.

Now the inclusion $C_q \subset GL_r\mathbb{F}_q$ induces a map $BC_q \to BGL\mathbb{F}_q^+$ and hence $H_*BC_q \to H_*BGL\mathbb{F}_q^+$. Since $BGL\mathbb{F}_q^+$ is a homotopy associative and commutative H-space, this map in turn extends to a ring homomorphism $S(\tilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$, where $S(-)$ denotes the symmetric algebra. Let $S'(-)$ denote the strict symmetric algebra, that is, the quotient of the symmetric algebra obtained by factoring out the ideal generated by all a^2 with $|a|$ odd. This refinement is only relevant when $l = 2$, since for l odd $S = S'$.

A.3.1 Homology ring of $F\psi^q$

We want to show the isomorphism in mod l homology by starting with calculating the mod l homology ring of $F\psi^q$ in the case $l|q-1$. The good thing of the case $l|q-1$ is that $B\mu = BC_q$ and μ is a cyclic group.

Note that we know the homology ring of U and BU:

$$H_*BU \cong \mathbb{Z}/l[c_1, c_2, \dots]$$

where $|c_i| = 2i$ and c_i is the i-th Chern class; and

$$H_*U \cong \mathbb{Z}/l\langle c_1, c_2, \dots\rangle$$

with $|x_i| = 2i - 1$.

The following proposition comes from Lemma 4.4, 4.5,[16]. I rewrote the proofs adding details of "delooping" to 4.4 and shortening the proof to 4.5.

Main point: From the above results of homology rings of U and BU, it's not hard to see that the spectral sequence associated to the fibre sequence $U \to F\psi^q \to BU$ collapses at E_2. To show that the maps in the spectral sequence are homomorphisms of algebras, we want to show that the fibre sequence can be double-delooped since the multiplication structure of the H-spaces are induced by the double loop structure. The Adams operation ψ^q does not commute with the Bott map but it does when we localize at l for l coprime to p.

Proposition A.2. *If $l|q-1$, then $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras.*

Proof. Consider the fibre sequence $U \to F\psi^q \to BU$ coming from the fibre sequence $F\psi^q \to BU \to BU$. Let $L : X \mapsto X_{(l)}$ denote the localization of X away from l. Let $\beta : BU \to \Omega_0^2 BU$ denote the Bott map, then we form a diagram where $h = L(\psi^q - 1)$

$$\begin{array}{ccc} BU & \xrightarrow{\psi^q - 1} & BU \\ \big\downarrow L & & \big\downarrow L \\ BU_{(l)} & \xrightarrow{h} & BU_{(l)} \\ \big\downarrow \beta & & \big\downarrow \beta \\ \Omega_0^2 BU_{(l)} & \xrightarrow{\Omega_0^2 h} & \Omega_0^2 BU_{(l)} \end{array}$$

Note that the Bott isomorphism $\beta : \tilde{K}X \xrightarrow{\cong} \tilde{K}(S^2 \wedge X)$ does not commute with the Adams operations. In fact, we have the formula $\psi^q(\beta a) = q\beta(\psi^q a)$ because $\beta a = b \times a$ with $b \in \tilde{K}S^2$ and ψ^q is multiplicative. Going around the right side gives $a \mapsto \beta\psi^q a - \beta a = 1/q\psi^q\beta a - \beta a$ and going around the left side gives $a \mapsto \psi^q\beta a - \beta a$. Moreover, multiplication by q is an equivalence for $BU_{(l)}$ since $(l, q) = 1$. We have that the above diagram homotopy commutes and h is equivalent to $\Omega_0^2 h$, a double-loop map. Now the fibre sequence associated to h is multiplicative meaning that h commutes with the H-space multiplication, thus the spectral sequence is a spectral sequence of Hopf algebras. Since L induces isomorphisms in mod l homologies, the same is true for $\psi^q - 1 : BU \to BU$.

Consider the fibre sequence mod l homology associated to $F\psi^q \to BU \to BU$. Since $\pi_1 BU$ is trivial, the E_2 term of this fibre sequence is $H_*U \otimes H_*BU$. We know that $H_*BU \cong \mathbb{Z}/l[c_1, c_2, \dots]$ where $|c_i| = 2i$ the Chern classes and $H_*U \cong \mathbb{Z}/l\langle x_1, x_2, \dots\rangle$ with $|x_i| = 2i-1$. Since the bidegree $(r, -r+1)$ of a differential map is odd for one and even for the other, by observing the degrees of nonzero $H_*U \otimes H_*BU$, the spectral sequence collapses at E_2. Thus the mod l homology $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras. □

A.3.2 Homology ring of $BGL\mathbb{F}_q^+$

Now we know the homology ring of $F\psi^q$. Recall at the beginning of this section the inclusion $C_q \subset GL_r\mathbb{F}_q$ induces a map $BC_q \to BGL\mathbb{F}_q^+$ and hence $H_*BC_q \to H_*BGL\mathbb{F}_q^+$. And in the case $l|q-1$ we have $B\mu \cong BC_q$. Since μ is cyclic, we can compute its group cohomology to be $H^*\mu = \mathbb{Z}/l[y] \otimes \mathbb{Z}/l\langle x \rangle$ with $|y| = 2$ and $|x| = 1$. Compared with the homology ring for $F\psi^q$, naturally we want to show that $B\mu$ is a generating complex.

Theorem A.2. *The natural map $S(\tilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$ induces an isomorphism $S'(\tilde{H}_*BC_q) \xrightarrow{\cong} H_*BGL\mathbb{F}_q^+$, here $S(V)$ denote the symmetric algebra of a graded vector space V and $S'(V)$ denote the strict symmetric algebra of V.*

Theorem A.2 (Theorem 4.6, [16])**.** *Let j be the composite $B\mu \to BGL\mathbb{F}_q^+ \to F\psi^q$. This map induces an isomorphism $S'(\tilde{H}_*B\mu) \xrightarrow{\cong} H_*F\psi^q$.*

In the case l odd, symmetric and strict symmetric algebra are the same and $BC_q = B\mu$ when $l|q-1$ (we've reduced to the case $l|q-1$), so from the following

$$S(\tilde{H}_*B\mu) \xrightarrow{\varphi} H_*BGL\mathbb{F}_q^+ \xrightarrow{\theta_*} H_*F\psi^q,$$

we have that $\theta_* : H_*BGL_n\mathbb{F}_q^+ \to H_*F\psi^q$ is an isomorphism since φ and the composite $\theta_*\varphi$ are both isomorphism from Theorem A.2.

The proofs of Theorem 3.1 and 3.2 require much more work and we refer to [16], [17] for the details. In the case $l = 2$, we need to show that $e_i = 0$ in $H_*BGL_n\mathbb{F}_q^+$, where e_i are generators of $H_{2i-1}B\mu$. This requires another ingeneous work to complete, see [Section 7, [16]] or [17].

We'd also like to mention that the calculations for homology rings are directly aimed at the computations of K-groups. If we carry out these calculations further, we can derive some further computations on the homology and cohomology of $GL_n\mathbb{F}_q$, see [17].

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