The Ring $k[x, y, u, v]/(xy - uv)$ is normal

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Let $A = k[x, y, u, v]/(xy - uv)$, is this ring normal (integrally closed in its field of fraction)? To check that the ring $k[x, y, u, v]/(xy-uv)$ is normal, we check Serre's criteria for normality:

Theorem 0.1. *Let A be Noetherian, then A is normal if and only if both of the following hold: (1) $A_{\mathfrak{p}}$ is a DVR for all primes $\mathfrak{p}$ of height 1. (2) For all $0 \neq x \in A$, if $\mathfrak{p} \in Ass(A/Ax)$ then $ht(\mathfrak{p}) = 1$.*

(1) Let $\mathfrak{p}$ be a prime ideal of height 1. Then $\mathfrak{p}$ doesn't contain all x, y, u, v otherwise $\mathfrak{p}$ is maximal with height > 1. We may assume $x \notin \mathfrak{p}$, then x becomes invertible in $A_{\mathfrak{p}}$. In $A_{\mathfrak{p}}$, $y = uv/x$ and we can write $A_{\mathfrak{p}} = k(x)[u, v]_{\bar{\mathfrak{p}}}$ where $\bar{\mathfrak{p}}$ is a prime ideal in $k(x)[u, v]$ of height 1. Note that $k(x)[u, v]$ is a domain, (0) is a prime ideal, thus any height 1 prime ideal in $k(x)[u, v]$ must be of the form $(f(u, v))$ for some nonzero irreducible polynomial f. So $\mathfrak{p}A_{\mathfrak{p}}$ is principal which verifies that $A_{\mathfrak{p}}$ is a DVR.

(2) Take any $0 \neq x \in A$. Since A is a domain, x is a non zero-divisor. Note that A is Cohen-Macaulay of dim 3 (check that a maximal A-sequence can be given by $\{x, y, u-1\}$). And $\dim A/xA = 2$, A/xA is Cohen-Macaulay of dimension 2.

Take any essential prime $\mathfrak{p}$ of xA, there is an injection

$$A/\mathfrak{p} \hookrightarrow A/xA.$$

Since p consists of zero divisor of A/xA, $\operatorname{depth} A/\mathfrak{p} = \operatorname{depth} A/xA = 2$. Moreover, $\dim A/\mathfrak{p} \leq \dim A/xA = 2$, thus $\dim A/\mathfrak{p} = 2$. Since A is Cohen-Macaulay, $ht(\mathfrak{p}) = \dim A - \dim A/\mathfrak{p} = 1$. This verifies condition (2). □