Counterexamples about Infinitely Generated Modules over Commutative Rings

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This post is about examples of some nice things that could fail for modules that are not finitely generated (or not finitely presented when the ring is not Noetherian).

1 Nakayama's Lemma (Krull's Intersection Theorem)

Proposition 1.1 (Krull's Intersection Theorem)**.** *Let A be a commutative ring, $M \neq 0$ a finitely generated A-module and I an ideal of A such that $M = IM$, then $\exists 0 \neq a \in I$ such that $(1-a)M = 0$. (Note that we do assume)*

Proof. Let $M = \langle x_1 \dots, x_n \rangle$. Since $IM = I$, there exits $a_{ij} \in I$ such that

$$\begin{aligned} x_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\dots \\ x_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

This is equivalent to

$$\begin{aligned} &(a_{11}-1)x_1 + \cdots + a_{1n}x_n = 0 \qquad (1) \\ &(a_{21}-1)x_1 + \cdots + a_{2n}x_n = 0 \qquad (2) \\ &\dots \\ &a_{n1}x_1 + \cdots + (a_{nn}-1)x_n = 0 \qquad (\text{n}) \end{aligned}$$

Let

$$A = \begin{bmatrix} a_{11}-1 & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-1 \end{bmatrix}.$$

Note that

equation (1)$\times$ the cofactor of $(a_{11}-1)+$ equation (2) $\times$ the cofactor of $a_{21} + \cdots +$ equation (n) $\times$ the cofactor of $a_{n1} = \det A \cdot x_1 = 0$.

Similarly, we have $\det A \cdot x_i = 0$ for all $1 \leq i \leq n$. Since $\det A = 1 - a$ for some $a \in I$, we have $\det A \cdot M = (1-a) \cdot M = 0$. □

Corollary 1.1 (Nakayama's lemma)**.** *Let A be a commutative ring, M a finitely generated A-module and $I \subset J(A)$ an ideal contained in the Jacobson radical of A, if $M = IM$, then $M = 0$.*

Proof. Since $1 - a$ is invertible for any $a \in I$. □

If M is not finitely generated then Krull's Intersection Theorem is not true.

Example 1.1. *Let $M = \mathbb{Q}$ as a $\mathbb{Z}$ module which is infinitely generated and $I = (p)$ for any prime $p \in \mathbb{Z}$, then $I\mathbb{Q} = \mathbb{Q}$ but there is no element $pk \in (p)$ such that $(1-pk)\mathbb{Q} = 0$ since $\mathbb{Q}$ has no zero divisor.*

2 Supp$M = V(\text{annM})$

Example 2.1. *Let* $M = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}/2^i\mathbb{Z}$ *as a* $\mathbb{Z}$*-module, then* $\operatorname{ann} M = 0$ *but* $\operatorname{supp} M = (2)$*.*

3 $\operatorname{Hom}_A(M, N) \otimes B = \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B)$ for B a flat A-module

Let $A \to B$ a flat ring map, For M a finitely presented A-module, N an A-module, we have $\operatorname{Hom}_R(M, N) \otimes B \cong \operatorname{Hom}_R(M \otimes_A B, N \otimes_A B)$.

Example 3.1. *Let* $A = \mathbb{Z}$*,* $S = \mathbb{Z} \setminus \{0\}$*,* $A \to B = \mathbb{Z}_S = \mathbb{Q}$ *the localization map. Let* $M = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ *an infinite direct sum of* $\mathbb{Z}$ *and* $N = \mathbb{Z}$*, then* $\prod_{i\in\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \ncong \prod_{i\in\mathbb{Z}} \mathbb{Q}$ *since* $(1, 1/2, 1/3, \dots) \in \prod_{i\in\mathbb{Z}} \mathbb{Q}$ *is not in the localization of* $\prod_{i\in\mathbb{Z}} \mathbb{Z}$ *at* S*.*

4 $M \otimes_R \hat{R} = \hat{M}$

Example 4.1. *Let* $R = \mathbb{Z}_p$ *be the ring of p-adic integers, i.e. the completion of* $\mathbb{Z}$ *at the maximal ideal* (p)*. Take* $M = \bigoplus \mathbb{Z}_p$ *as an infinite direct sum of* $\mathbb{Z}_p$*. Then* $R = \hat{R}$*, thus* $M \otimes_R \hat{R} = M$ *but* M *is not complete since the Cauchy sequence* $a_n = (p, p^2, \dots, p^n, 0, 0, \dots)$ *does not converge in* M*.*

Example 4.2. *Some other examples in this post: How to find a non-surjective and non-injective tensor products of the formal completion?*

5 Finitely Generated Nil Ideal is Nilpotent

If given a finitely generated nil ideal, then it's clearly nilpotent.

Let $N(R)$ be the nilradical of a commutative ring R, then $N(R)$ is a nil ideal, i.e. every element in $N(R)$ is nilpotent. But the ideal needs not be nilpotent in general, i.e. there is no n such that $N(R)^n = 0$.

Example 5.1 (Zassenhaus's Example)**.** *Let* F *be a field,* I *the open interval* $(0, 1)$ *and* R *a vector space over* F *with basis* $\{x_i\}_{i\in I}$*. Define a multiplication on* F *by extending the following product of basis elements*

$$x_i x_j = \begin{cases} x_{i+j} & \text{if } i + j < 1 \\ 0 & \text{if } i + j \geq 1 \end{cases}.$$

Every element of R *can be written uniquely as a finite sum* $\Sigma a_i x_i$*, hence every element in* R *is nilpotent, so* $N(R) = R$*. But* R *is not nilpotent.*

Remark. *In a commutative ring* R*, the nilradical* $N(R)$ *equals the set of all nilpotent elements but it needs not be true if* R *is not commutative.*

For example, let R *be the ring of* 2×2 *matrices over* $\mathbb{Q}$*. Then* R *has only two ideal* (0) *and* R*. So* $N(R) = 0$ *but* $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0$*.*

6 Finitely Presented Flat Module is Projective

Proposition 6.1. *Every finitely presented free R-module is projective.*

Proof. To show that M is projective, we want to show that $\mathrm{Hom}_R(-, M)$ is exact. Let $B \to C$ be surjective, let $(-)^* = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ denote the Pontrjagin dual, then $C^* \to B^*$ is injective.

$$\begin{array}{ccc} (C^*) \otimes_R M & \longrightarrow & (B^*) \otimes_R M \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Hom}(M, C)^* & \longrightarrow & \mathrm{Hom}(M, B)^* \end{array}$$

Since M is flat, we have that the top arrow is an injection. And the vertical arrows are isomorphisms for finitely presented M (since it holds if M is finite free). This implies the bottom arrow is injective, hence the map $\mathrm{Hom}(M, C) \to \mathrm{Hom}(M, B)$ is surjective. □

Example 6.1. $\mathbb{Q}$ *as a* $\mathbb{Z}$*-module is not projective. Otherwise, by Structure Theorem over PID,* $\mathbb{Q}$ *must be free hence a direct sum of* $\mathbb{Z}$ *which is not possible.*