Depth and Dimension of the Fibre of a Local Ring Homomorphism

Likun Xie

March 19, 2022

An important information of a homomorphism of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is its fibre $S/\mathfrak{n}S$. We introduce two main theorems that relate the depth and dimensions and some corollaries. Finally we apply them to show that the polynomial ring or formal power series ring over a Cohen-Macaulay ring is Cohen-Macaulay. See also Theorem A.11 in [1] and Theorem 23.2 in [2].

Theorem 0.1. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of noetherian local rings, i.e. $f(\mathfrak{m}) \subset \mathfrak{n}$.*

Suppose B is A-flat via f, then

$$\operatorname{depth}_{\mathfrak{n}} B = \operatorname{depth}_{\mathfrak{m}} A + \operatorname{depth}_{\mathfrak{n}} B/\mathfrak{m}B.$$

Proof. We first prove the following claim:

Claim. If $x \in B$ is a non-zero-divisor on $B/\mathfrak{m}B$, then x is a non-zero-divisor on B and B/xB is A-flat.

Proof of the claim. Let $y \in B$ be such that $xy = 0$. Suppose $y \neq 0$. Since B is noetherian local, $\bigcap \mathfrak{m}^t B \subset \bigcap \mathfrak{n}^t = 0$. Thus $y \notin \mathfrak{m}^t B$ for some $t > 0$, x is a zero-divisor on $B/\mathfrak{m}^t B$.

Consider $A/\mathfrak{m}^t$, since $l(A/\mathfrak{m}^t) < \infty$, there is a finite filtration

$$A/\mathfrak{m}^t \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_r = (0)$$

where $A_i/A_{i+1} = A/\mathfrak{m}$.

Apply $- \otimes_A B$, note that B is A flat,

$$B/\mathfrak{m}^t B \supsetneq A_1 \otimes B \supsetneq A_2 \times B \supsetneq \cdots \supsetneq A_r \otimes B = (0)$$

such that $A_i \otimes B/A_{i+1} \otimes B \cong A_i/A_{i+1} \otimes B = A/\mathfrak{m} \otimes B = B/\mathfrak{m}B$.

Since x is a non-zero-divisor on $B/\mathfrak{m}B$, x is a non-zero-divisor on all $B/\mathfrak{m}^t B$, $\forall t$, a contradiction.

To show that B/xB is flat over A, consider $0 \to M_1 \to M_2 \to M_3 \to 0$ a short exact sequence of finitely generated A-modules, we have the following commutative diagrams

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 \otimes_A B & \longrightarrow & M_2 \otimes_A B & \longrightarrow & M_3 \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow x & & \downarrow x & & \downarrow x & & \\
0 & \longrightarrow & M_1 \otimes_A B & \longrightarrow & M_2 \otimes_A B & \longrightarrow & M_3 \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M_1 \otimes_A B/xB & \longrightarrow & M_2 \otimes_A B/xB & \longrightarrow & M_3 \otimes_A B/xB & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all columns are exact since x is a non-zero-divisor on $M_i \otimes B$. The first two rows are exact since B is A-flat, hence the bottom row is also exact by an argument of diagram chasing. $\square$

Let's continue with the proof by induction. Suppose depth $= 0, \operatorname{depth} M/\mathfrak{m}B = 0$, then we have $A/\mathfrak{m} \xhookrightarrow{x} A$, $B/\mathfrak{n} \xhookrightarrow{\bar{y}} B/\mathfrak{m}B$. Since B is A-flat, $B/\mathfrak{m}B = A/\mathfrak{m} \otimes_A B \xhookrightarrow{x} B$. Thus $B/\mathfrak{n}B \xhookrightarrow{\bar{y}x} B$, $\mathfrak{n} \in \operatorname{Ass}_B(B)$, $\operatorname{depth} B = 0$.

If $\operatorname{depth} A > 0$, then there exists $x \in \mathfrak{m} \subset A$ a non-zero-divisor on A. Thus $A/xA \to B/xB = \bar{B}$ is flat and $\bar{B}/\mathfrak{m}\bar{B} = B/\mathfrak{m}B$. By induction on $\operatorname{depth} A$,

$$\operatorname{depth} B/xB = \operatorname{depth} A/xA + \operatorname{depth} \bar{B}/\mathfrak{m}\bar{B}.$$

Note that x is also a non-zero-divisor on B since B is A-flat, so by induction

$$\operatorname{depth} B - 1 = \operatorname{depth} A - 1 + \operatorname{depth} B/\mathfrak{m}B.$$

If $\operatorname{depth} B/\mathfrak{m}B > 0$, then there exists $y \in B/\mathfrak{m}B$ a non-zero-divisor on $B/\mathfrak{m}B$. By the claim, y is a non-zero-divisor on B and B/yB is A-flat.

By induction,

$$\operatorname{depth} B - 1 = \operatorname{depth} B/yB = \operatorname{depth} A + \operatorname{depth} \bar{B}/\mathfrak{m}\bar{B} = \operatorname{depth} A + \operatorname{depth} B/\mathfrak{m}B - 1.$$

$\square$

By a similar argument as above, one can show the following for which we can show the claim that for $x \in B$ a non-zero-divisor on $A/\mathfrak{m}N$, x is a non-zero-divisor on $M \otimes_A N$ and N/xN is flat over A.

Theorem 0.2. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of local rings, i.e. $f(\mathfrak{m}) \subset \mathfrak{n}$. Let M be a finitely generated A-module, N a finitely generated B-module and N is A-flat.*

$$\operatorname{depth}_{\mathfrak{n}} M \otimes_A N = \operatorname{depth}_{\mathfrak{m}} M + \operatorname{depth}_{\mathfrak{n}} N/\mathfrak{m}N.$$

We have the following theorem that relates the dimensions.

Theorem 0.3. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of local rings, i.e. $f(\mathfrak{m}) \subset \mathfrak{n}$. Let M be a finitely generated A-module, N a finitely generated B-module and N is A-flat. Then*

$$\dim_{\mathfrak{n}} M \otimes_A N = \dim_{\mathfrak{m}} M + \dim_{\mathfrak{n}} N/\mathfrak{m}N.$$

The crucial point of the proof of Theorem 0.3 is the following proposition relating associated primes. Note that in the following, A, B need not be local but are noetherian, and M needs not be finitely generated.

Proposition 0.1. *Let $f : A \to B$ be a ring homomorphism (not necessarily local), M an A-module (not necessarily finitely generated), N a finitely generated B-module and is A-flat. then*

$$\operatorname{Ass}_B(M \otimes_A N) = \bigcup_{p \in \operatorname{Ass}_A(M), N \neq pN} \operatorname{Ass}_B(N/pN).$$

Proof. First we show that RHS $\subset$ LHS. Let $p \in \mathrm{Ass}_A(M)$, then $A/p \hookrightarrow M$. Since N is A-flat, $N/pN = A/p \otimes_A N \hookrightarrow M \otimes_A N$, thus $\mathrm{Ass}_B(N/pN) \subset \mathrm{Ass}_B(M \otimes_A N)$.

Next we show LHS $\subset$ RHS.

Case 1. Suppose M is p-coprimary, i.e. $\mathrm{Ass}_A(M) = \{p\}$. Then every element of M is killed by a power of p. Hence every element of $M \otimes_A N$ is killed by a power of p. Thus for any $Q \in \mathrm{Ass}_B(M \otimes_A N), Q \supset pB$.

Let $Q \in \mathrm{Ass}_B(M \otimes_A N)$, $x \in A \setminus p$, then $M \xrightarrow{x} M$ is injective, so $M \otimes_A N \xrightarrow{x} M \otimes_A N$ is injective since N is A-flat.

So $f^{-1}(Q) \subset p$, hence $f^{-1}(Q) = p$.

Now take a prime filtration of M:

$$M = M_0 \supseteqq M_1 \supsetneqq \cdots \supsetneqq M_n = (0)$$

where $M_i/M_{i+1} \cong A/p_i$ and $\mathrm{Ass}_A M = \{p\} \subset \{p_0, \dots p_{n-1}\}$.

Since N is A-flat, we have a filtration

$$M \otimes_A N = M_0 \otimes_A N \supseteqq M_1 \otimes_A N \supsetneqq \cdots \supsetneqq M_n \otimes_A N = (0)$$

such that $M_i \otimes_A N/M_{i+1} \otimes_A N \cong M_i/M_{i+1} \otimes_A N \cong N/p_iN$. Thus $\mathrm{Ass}_B(M \otimes_A N) \subset \bigcup_{i=0}^{n-1} \mathrm{Ass}_B(N/p_iN)$. Since for any $P \in \mathrm{Ass}_B(N/p_iN))$, $f^{-1}(P) \supseteq p_i \supseteq p$, but from above $f^{-1}\mathrm{Ass}_B(M \otimes_A N) = \{p\}$, we must have $\mathrm{Ass}_B(M \otimes_A N) \subset \mathrm{Ass}_B(N/pN)$.

Case 2. Suppose M is finitely generated and let $0 = N_1 \cap N_2 \cap \dots N_t$ be a primary decomposition, where each N_i is p_i-primary.

Then $M \hookrightarrow \bigoplus M/N_i$, hence $M \otimes_A N \hookrightarrow \bigoplus(M/N_i \otimes_A N)$ as N is A-flat.

Thus $\mathrm{Ass}_B(M \otimes_A N) \subseteq \bigcup \mathrm{Ass}_B(M/N_i \otimes_A N) \subseteq$ (case 1) $\bigcup_{p_i \in \mathrm{Ass}_A M} \mathrm{Ass}_B(N/p_iN)$.

Case 3. M is not necessarily finitely generated, $M = \varinjlim M_\lambda$, M_λ is finitely generated.

Then $M \otimes_A N = \varinjlim M_\lambda \otimes_A N$. If $Q \in \mathrm{Ass}_B(M \otimes_A N)$, then $Q \in \mathrm{Ass}_B(M_\lambda \otimes_A N)$ for some λ then apply case 2. ☐

Proof of Theorem 0.3. Suppose $\dim_{\mathfrak{m}} M = 0$, then $\mathrm{Ass}_A M = \{\mathfrak{m}\}$, by Proposition 0.1, $\mathrm{Ass}_B(M \otimes_A N) = \mathrm{Ass}_B(N/\mathfrak{m}N)$, thus $\dim_{\mathfrak{n}} M \otimes_A N = \dim_{\mathfrak{n}} N/\mathfrak{m}N$.

Suppose $\dim_{\mathfrak{m}} M > 0$, let $x \in A$ be a non-zero-divisor on M, then since N is A-flat, x is also a non-zero-divisor on $M \otimes_A N$. By induction,

$$\dim_{\mathfrak{n}}(M \otimes_A N)/x = \dim_{\mathfrak{m}} M/x + \dim_{\mathfrak{n}} N/\mathfrak{m}N,$$

moreover, $\dim_{\mathfrak{n}} M \otimes_A N - 1 = \dim_{\mathfrak{n}}(M \otimes_A)N/x$, $\dim_{\mathfrak{m}} M - 1 = \dim_{\mathfrak{m}} M/x$ hence the theorem. ☐

Corollary 0.1. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of local rings, i.e. $f(\mathfrak{m}) \subset \mathfrak{n}$. Let M be a finitely generated A-module, N a finitely generated B-module and N is A-flat. Then $M \otimes_A N$ is Cohen-Macaulay $\iff$ M is Cohen-Macaulay and $N/\mathfrak{m}N$ is Cohen-Macaulay.*

Corollary 0.2. *Let A be a local ring, M a finitely generated A-module. Then M is Cohen-Macaulay $\iff$ $\hat{M}$ is Cohen-Macaulay.*

Proof. Since $A \to \hat{A}$ is flat, $\hat{M} = M \otimes_A \hat{A}$, apply the previous theorems. ☐

Corollary 0.3. *Let $f : R \to S$ be a faithfully flat homomorphism of noetherian local rings. Then S is Cohen-Macaulay $\iff$ R is Cohen-Macaulay, and for any maximal ideal Q of S, $q = Q \cap R$, $\kappa(p) \otimes_R S = R_p/pR_p \otimes_R S = S_p/pS_p$ (the fibre over p in* $\mathrm{Spec}\, S$*) is Cohen-Macaulay.*

Applications

1 Let A be a noetherian ring, if A is Cohen-Macaulay, then $A[x_1, \dots x_n]$ is Cohen-Macaulay.

Proof. For any maximal ideal Q of $A[x_1, \dots, x_n]$ and $q = Q \cap A$, $\kappa(p) \otimes_A B = \kappa(p)[x_1, \dots, x_n]$ is regular hence Cohen-Macaulay. Then apply Corollary 0.3. □

2 Let A be a noetherian ring, if A is Cohen-Macaulay, then $A[[x_1, \dots x_n]]$ is Cohen-Macaulay.

Note that for $A \to K$ where K is a field, it's not necessarily true that $K \otimes_A A[[x_1, \dots, x_n]] \cong K[[x_1, \dots, x_n]]$. This is only true when K is module finite over A.

Proof. To prove 2, enough to prove for $A \to A[[x]]$. Let $\mathfrak{n}$ be a maximal ideal of $A[[x]]$. Then $x \in \mathfrak{n}$ since $A[[x]]$ is the completion of $A[x]$ is the xA-adic topology. Thus $A[[x]]/\mathfrak{n} = A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A. Since $A/\mathfrak{m} \otimes_A A[[x]] = A/m[[x]]$ is Cohen-Macaulayand A is Cohen-Macaulay, we have $A[[x]]$ is Cohen-Macaulay/ □

References

[1] Bruns, Winfried, and H. Jürgen Herzog. *Cohen-macaulay rings.* No. 39. Cambridge university press, 1998.

[2] Matsumura, Hideyuki. *Commutative ring theory.* No. 8. Cambridge university press, 1989.