Hartshorne's Connectedness Theorem:
Connectedness of Punctured Spectrum of a Local Ring of Depth ≥ 2

Likun Xie
March 13, 2022

The following theorem is Hartshorne's Connectedness Theorem. From this we see that the punctured spectrum of a noetherian local ring A is connected if $\operatorname{depth} A \geq 2$.

Theorem 0.1. *Let $(A, \mathfrak{m})$ be a noetherian local ring,* $\operatorname{depth} A \geq 2$*. Suppose there exists ideals $I, J \subset \mathfrak{m}$ such that $I + J$ is $\mathfrak{m}$-primary and $I \cap J$ is nilpotent. Then either I or J is nilpotent.*

Proof. Given $I + J$ $\mathfrak{m}$-primary and $I \cap J$ nilpotent, by taking powers of I and J, one can show that for some r, $I^r + J^r$ is $\mathfrak{m}$-primary and $I^r \cap J^r = (0)$, i.e. we can assume $I \cap J = (0)$ and $I + J$ is $\mathfrak{m}$-primary.

(To see this, since $I \cap J$ is nilpotent, there exists some $t > 0$ such that $(I \cap J)^t = 0$. By Artin-Rees, there exists i such that $I^{n+i} \cap J^t = I^n(I^i \cap J^t) \subset I^n J^t$. Take $n > t$, $I^n \cap J^t \subset (IJ)^t \subset (I \cap J)^t = 0$.)

Now since $\operatorname{depth} A > 2$ and $I + J$ is $\mathfrak{m}$ primary, we can choose $a, a' \in I$, $b, b' \in J$ such that $a + b, a' + b'$ form an M-sequence.

Then $a(a' + b') - a'(a + b) = 0$. Regularity implies that $a = \lambda(a + b)$. If λ is a unit, then a is a non-zero-divisor on A, but $aJ = 0$ $(IJ = 0)$, thus $J = 0$.

If λ is not a unit, then $1 - \lambda$ is a unit. Thus $I \ni a(1 - \lambda) = \lambda b \in J$. Since $I \cap J = 0$, $a = 0 \Rightarrow b$ is a non-zero-divisor on A. But $bI = 0 \Rightarrow I = 0$. $\square$

Remark. *This implies that given the condition in the theorem,* $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$ *is connected.*

Proof. Suppose $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$ is not connected, then $\operatorname{spec} A \setminus \{\mathfrak{m}\} = V(I) \cup V(J)$ and $V(I) \cap V(J) = \emptyset$.

Then $I \cap J$ is nilpotent and $I + J$ is $\mathfrak{m}$-primary. Then either I or J is nilpotent, thus $\operatorname{spec} A \setminus \{\mathfrak{m}\} = V(I)$ or $V(J)$. $\square$

Let's apply the theorem to the ring $R = \mathbb{C}[x, y, w, v]/(x, y) \cap (w, v)$ to show that R is not Cohen-Macaulay.

Note that there is a chain of prime ideals as following

$$(x, y) \subset (x, y, w) \subset (x, y, w, v).$$

So $\dim R \geq 2$, if R is Cohen-Macaulay, then $\operatorname{depth} R \geq 2$. Applying the Connectedness Theorem to $I = (x, y)$ and $J = (w, v)$ gives a contradiction.