The Grothendieck Ring of the Category of Endomorphisms

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Abstract

The aim of this note is to compute the Grothendieck group K_0 of the category of endomorphisms. This computation mostly plays with linear algebra. The main result is that in K_0, every endomorphism $f : P \to P$ is uniquely characterized by P and its characteristic polynomial $\lambda_t(f)$. This computation was due to [2]. We will explain how to think about this computation, the reason for certain constructions and the "diagonalization" in this computation.

Let A be a commutative ring with identity element 1. We have the following categories of A-modules:

$$P(A) \subset H(A) \subset M(A)$$

where $P(A)$ is the category of finitely generated projective A-modules, $M(A)$ is the category of finitely generated A-modules and $H(A)$ is the full subcategories consisting of those modules with finite projective resolutions.

Let End$M(A)$ denote the category where the objects are endomorphisms $f : M \to M$ with M in $M(A)$ and the morphisms are commutative diagrams.

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ \downarrow f & & \downarrow g \\ M & \xrightarrow{u} & N \end{array} .$$

Analogously, $EndP(A)$ and $EndH(A)$ are defined. The Grothendieck group $K_0(\text{End}M(A))$ is the free abelian group generated by all isomorphism classes of objects in End$M(A)$ modulo the subgroup generated by all $[f] - [f'] - [f'']$ where

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array} \tag{1}$$

is exact and commutative. Similarly, we define $K_0(EndP(A))$ and $K_0(EndH(A))$.

1 The characteristic polynomial

We have two ring homomorphisms

$$K_0(EndP(A)) \underset{\psi}{\overset{\pi}{\rightleftarrows}} K_0(A) \tag{2}$$

where $\pi[P, f] = [P]$ (forget f) and $\psi[P] = [P, 0]$. Clearly $\pi \circ \psi = Id$. So we have

$$K_0(EndP(A)) \cong K_0(A) \times \tilde{K_0}(EndP(A))$$

as rings.

Now given an A-linear map $f : P \to P$. We want to define *the characteristic polynomial* $\lambda_t(f)$ of f by the formula

$$\lambda_t(f) = \Sigma_{i\geq 0} Tr(\Lambda^i f)t^i$$

where Tr denotes the trace. Another way to get $\lambda_t(f)$ is the following. There is an A-module Q such that $P \oplus Q \cong A^n$ is free. Extend f to $P \oplus Q$ by the zero map and define

$$\lambda_t(f) = det(1 + t(f \oplus 0_Q)).$$

This is independent of the choice of Q (Why? If we take $P \oplus Q = A^n$ and $P \oplus Q' = A^m$ $(n < m)$, then $det(1 + t(f \oplus 0_Q))$ and $det(1 + t(f \oplus 0_{Q'}))$ are the same.)

To see that this two definitions are the same, let's look at a matrix A. We have $det(1+tA) = \Sigma t^i C_i$, $C_i = \Sigma_{|J|=i} det A_J$ where the sum ranges over the determinant of all minors of A of size i and $J \subseteq \{1, 2, \ldots, n\}$ of size i. We want to show that $tr\Lambda^m A = \Sigma_{|J|=m} det A_J$. Let I be a subset of $\{1, 2, \ldots, n\}$, we write $e_I = \Lambda_{i\in J} e_i$, we can check that $\Lambda^m A e_J = det A_J e_J + \Sigma_{I\neq J} \alpha_I e_I$. To see this, we can apply $\Lambda^m A$ to e_J:

$$\Lambda^m A(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) = A(e_{i_1}) \wedge A(e_{i_2}) \wedge \cdots \wedge A(e_{i_m}).$$

We can expand each term $A(e_{i_r})$ and pull out the term involving only $e_{i_1}, \ldots e_{i_m}$, we have the above is equal to

$$det(A_J) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} + \text{other terms}.$$

Thus

$$tr\Lambda^m A = \text{the sum of diagonal entries} = \Sigma_{|J|=m} det A_J.$$

The main point that we want to show is that the every object in $K_0(EndP(A))$ is determined by the underlying projective module and its characteristic polynomial $\lambda_t(f)$.

How do we do that? First let's check that give an exact sequence of endomorphism 1, we have $\lambda_t(f) = \lambda_t(f')\lambda_t(f'')$. Suppose we have Q', Q'' with $P' \oplus Q' = A^n$ and $P'' \oplus Q'' = A^m$, then we have $P \oplus Q' \oplus Q'' = A^{n+m}$. Extend the homomorphisms in 1, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^n & \longrightarrow & R^{n+m} & \longrightarrow & R^m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (f',0)} & & \downarrow{\scriptstyle \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}} & & \downarrow{\scriptstyle (f'',0)} & & \\
0 & \longrightarrow & R^n & \longrightarrow & R^{n+m} & \longrightarrow & R^m & \longrightarrow & 0
\end{array}$$

where A, B are the matrix representing $(f', 0)$ and $(f'', 0)$ respectively and C is some matrix. Thus we have

$$det(1 + t\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}) = det(1 + t(f', 0)) \cdot det(1 + t(f'', 0)).$$

So we have that $\lambda_t(f) = \lambda_f(f')\lambda_t(f'')$. It follows that λ_t is defined on $\tilde{K_0}(EndP(A))$ by $\lambda_t([f] - [g]) = \lambda_t(f)/\lambda_t(g)$.

Hence, we have a group homomorphism

$$\lambda_t : \tilde{K_0}(EndP(A)) \to \tilde{A_0} = \{\frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m}; a_t, b_t \in A\}$$

where $\tilde{A_0}$ is a group under multiplication.

2 λ_t is surjective

Note that λ_t is surjective. To see this, note that every polynomial of the form $1+a_1t+\dots a_nt^n$ has the matrix

$$\begin{pmatrix} 0 & 0 & & 0 & 0 & -a_n \\ 1 & 0 & & 0 & 0 & a_{n-1} \\ . & & & . & . & . \\ . & & & . & . & . \\ . & & & . & . & . \\ 0 & 0 & & 0 & 0 & \pm a_2 \\ 0 & 0 & & 0 & 0 & \pm a_1 \end{pmatrix}$$

whose characteristic polynomial is the polynomial given. Such matrix defines an endomorphism of a free module of rank n. And recall that every element in $K_0(EndP(A))$ can be written as a difference of two elements in $EndP(A)$. Thus λ_t is surjective. (Note that there can be different way of defining characteristic polynomial of a matrix.)

To define a $*$-multiplication for $\tilde{A_0}$ we use the formula

$$\lambda_t(f) * \lambda_t(g) = \lambda_t(f \otimes g).$$

Then $\tilde{A_0}$ becomes a commutative ring. The multiplication $*$ is rather complicated, to indicate the flovour of the $*$-multiplication, we write out the formula

$$(1+a_1t+a_2t^2+\cdots+a_nt^n)*(1+b_1t+b_2t^2+\dots b_mt^m) = 1+a_1b_1t+(a_1^2b_2+a_2b_1^2-2a_2b_2)t^2+\cdots+a_n^mb_m^n.$$

The ring $\tilde{A_0}$ is isomorphic to a certain subring of the Witt ring $W(A)$ of A.

Theorem 2.1. *Let A be any commutative ring. Then the map*

$$\lambda_t : \tilde{K_0}(EndP(A)) \to \tilde{A_0}$$

is a ring isomorphism.

3 Identifying an endomorphism as an $A[t]$-module

Before proving this theorem, let's contemplate for a moment how we can prove this theorem. We have already shown that the map λ_t is surjective. To show that the map is injective, we want to show that different endomorphisms in $\tilde{K_0}(EndP(A))$ with the same characteristic polynomial must be isomorphic.

We are going to look at the category of $A[t]$ modules. If $f : P \to P$ is A-linear, then P can be considered as an $A[t]$-module where the $A[t]$-action is defined by $t \cdot x = f(x)$ for all $x \in P$. Conversely, any $A[t]$-module P corresponds to an endomorphism $P \to P$. The commutative exact diagram of A-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \xrightarrow{u} & P & \xrightarrow{v} & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & P' & \xrightarrow{u} & P & \xrightarrow{v} & P'' & \longrightarrow & 0 \end{array}$$

is the same thing as a short exact sequence

$$0 \to P' \to P \to P'' \to 0$$

as $A[t]$-modules. Hence we can identify $EndP(A)$ with the subcategory of $A[t]$-modules P such that P considered as A-module is in $P(A)$.

First, we can reduce to the case when P is A-free. Given $f : P \to P$ with $P \in P(A)$ finitely generated projective, there exists $Q \in P(A)$ such that $P \oplus Q = F$ is free. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f\oplus 0 & & \downarrow 0 & & \\ 0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \end{array}.$$

Hence in $K_0(EndP(A))$ we have $[f] = [f \oplus 0] - [0]$ where $[0] = [Q, 0]$ can be identified with $[Q]$ in $K_0(A)$ via the splitting map 2, thus $[P, f] = [F, f \oplus 0] \bmod K_0(A)$.

Thus we may assume that $P = A^n$ and $f : P \to P$ is given by an $n \times n$ matrix $f = (a_{ij})$. Then we have the following "characteristic sequence of f", exact in the category of $A[t]$-modules [[1], P.630]

$$0 \longrightarrow B^n \xrightarrow{tI-(a_{ij})} B^n \xrightarrow{\pi} P \longrightarrow 0 \tag{3}$$

where we let $B = A[t]$, $P = A^n$ and π is given by $\pi(\Sigma_{i=0}^k u_i t^i) = \Sigma_{i=0}^k f^i(u_i)$.

4 λ_t is injective

4.1 Contemplating the case for $A[t]$ a PID

View an endomorphism as an $A[t]$-module, let's think for a moment why different endomorphisms can have different characteristic polynomial. Suppose we have $A[t]$ as a PID, then we can apply the structure theorem on an $A[t]$-module V to write its invariant factors decomposition $V = \bigoplus_{i=1}^k A[t]/(f_i)$ with $f_1|f_2|\dots f_k$. Then we have that the characteristic polynomial of V is given by $f = f_1 \cdot f_2 \cdot \dots f_k$. Different invariant factors can give the same characteristic polynomial. But in K_0, we have the short exact sequence

$$0 \to A[t]/(f) \xrightarrow{g} A[t]/(fg) \to A[t]/(g) \to 0$$

for prime f and g. Thus we can write $f = \prod p_i^{k_i}$ as a product of prime factors and every endomorphism with characteristic polynomial f is isomorphic to the sum of $A[f]/p_i$'s.

4.2 First attempt for diagonalization

Having this in mind, we contemplate the general case. Given a general matrix, we speculate that we can identify this matrix with some diagonalized form the product of whose diagonal entries give the characteristic polynomial.

First try: Note that we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$$

Let $P = A^n$ and $B = A[t]$. In particular, we can take the above block matrix A to be 1×1, i.e. A is the $(1, 1)$ entry in the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and D is an $(n-1) \times (n-1)$ matrix.

We take $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be the matrix $tI - (a_{ij})$ as in the exact sequence **??**. Now from the exact sequence below, $[P] = [P']$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} & B^n & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow[\begin{pmatrix} A-BD^{-1}C & B \\ 0 & D \end{pmatrix}]{} & B^n & \longrightarrow & P' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4}$$

Moreover, from the exact sequence below, we have $[P'] = [B/a] + [P'']$ where $a = A - BD^{-1}C$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{A-BD^{-1}C} & B & \longrightarrow & B/(a) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow{g'} & B^n & \longrightarrow & P' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{n-1} & \xrightarrow{D} & B^{n-1} & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5}$$

where g' is given by $\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$.

Continuing this diagonalization process, we can have $[P] = [B/(a_1)] + [B/(a_2)] + \cdots + [B/(a_n)]$ where a_i are the diagonal entries of the diagonalization of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which is determined by the characteristic polynomial of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

This is the idea except that we can not assume that every matrix with entries in $A[t]$ has an inverse.

4.2.1 Modification of the first attempt

To fix this, we don't want to use matrix involving inverse of some matrices. We modify the matrices in 4 as follow

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow{g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}} & B^n & \xrightarrow{\pi} & P & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h=\begin{pmatrix} I & 0 \\ -C & B_{11} I_{n-1} \end{pmatrix}} & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow[g'=\begin{pmatrix} B_{11} & B \\ 0 & B_{11}D-CB \end{pmatrix}]{} & B^n & \xrightarrow{\pi'} & P' & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\
 & & & & C_1 & \xrightarrow{\cong} & C_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\; .
$$

One checks that $C_1 = (B/(A))^{n-1}$ and $hg = g'$ [see [2] P.382 for details]. Thus in K_0, we have $[P] = [P'] - (n-1)[B/(A)]$. Next, we modify 5 for reducing the size of P' to $n-1$. This is done in the following:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{A} & B & \longrightarrow & B/(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^n & \xrightarrow{g'} & B^n & \longrightarrow & P' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{n-1} & \xrightarrow{B_{11}D-CB} & B^{n-1} & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\; .
$$

From this, we have $[P'] = [P''] + [B/(A)]$. Inductively continuing this "diagonalization" process, we can write $[P]$ as a finite sum of $[B]/(p_i)$ for some monic polynomials p_i where p_i's are uniquely determined by the matrix's characteristic polynomial (Remember in K_0, we can always split a polynomial into its prime factors to write $[B/(fg)] = [B/(f)] + [B/(g)]$.) For more details, see [2].

References

[1] Bass, H. (1968). Algebraic K-theory. Mathematics Lecture Note Series.

[2] Almkvist, G. (1974). The Grothendieck ring of the category of endomorphisms. *Journal of Algebra*, 28(3), 375-388.