Brauer Lifting in Algebraic K-Theory of Finite Fields

Likun Xie

April 28, 2022

Abstract

Quillen's calculation of algebraic K-theory of finite fields incorporates a wide range of techniques from group cohomology to representation theory. This exposition focuses on the proof of Green's theoerm: For any finite group G and any representation of G over $\mathbb{F}_q$ (q is a power of some prime p), one can construct a character called the Brauer character which is a virtual complex character. We will also introduce its application in algebraic K-theory. The Brauer character of the n-dimensional standard representation of $GL_n(\mathbb{F}_q)$ is a virtual complex character, thus it induces a map $GL_n(\mathbb{F}_q) \to GL(\mathbb{C})$, hence a map $BGL\mathbb{F}_q^+ \to BGL(\mathbb{C}) \cong BU$. Taking the n-th homotopy group gives a map $\theta : K_n(\mathbb{F}_q) \to \pi_n BU$. This map is a key construction that allows one to identify $K_n(\mathbb{F}_q)$ with the homotopy groups of a better understood space.

Contents

1 Brauer Induction Theorem **1**
1.1 Representation Ring 1
1.2 Brauer Induction and Green's Theorem 3

2 K-theory of Finite Fields and Brauer Lifting **5**
2.1 K-theory of Finite Fields 5
2.2 Brauer Lifting 6

1 Brauer Induction Theorem

In this section, we introduce Brauer Induction Theorem and prove Green's theorem that the Brauer character of a representation over $\mathbb{F}_q$ is a virtual complex character. The reference is [3] and [1].

1.1 Representation Ring

Definition 1.1. *Let G be a finite group, F a field. Let $\mathrm{Rep}(G, F)$ be the set of isomorphism classes of finite-dimensional FG-modules and we set $[V] \sim [V'] + [V'']$ whenever there is a short exact sequence*

$$0 \to V' \to V \to V'' \to 0.$$

*The **representation ring** $R_F G$ is defined to be $\mathbb{Z}\,\mathrm{Rep}(G, F)/\sim$, the free group generated by $\mathrm{Rep}(G, F)$ modulo the relation $\sim$. An element of $R_F G$ is called a **virtual character**.*

If char$F \nmid |G|$, all such sequences split and $R_F G$ is the group completion of the monoid $\mathrm{Rep}(G, F)$. But this is not true in general.

The ring structure is tensor product of representations with G acting diagonally. A homomorphism $\phi : H \to G$ induces a ring homomorphism $\phi^* : R_F(G) \to R_F(H)$. For $i : H \subset G$ an inclusion, write Res_H^G for i^*.

If $H \subset G$ and W is an FH-module, the **induced module** is $FG \otimes_{FH} W$. Since FG is a free hence flat FH-module, it follows that induction is well-defined on $R_F H$ and yields a homomorphism of groups $\mathrm{Ind}_H^G : R_F H \to R_F G$.

Let K be another subgroup of G. Choose a set of representatives S for the (H, K) double cosets of G; this means that G is the disjoint union of the KsH for $s \in S$. For $s \in S$, let $H_s = sHs^{-1} \cap K$ which is a subgroup of K. Let $\rho : H \to GL(W)$ be a represetation of H, if we set

$$\rho_s(x) = \rho(s^{-1}xs)$$

for $x \in H_s$, we obtain a a representation of H_s, denoted as W_s. Since H_s is a subgroup of K, the induced representation $\mathrm{Ind}_{H_s}^K(W_s)$ is defined.

Proposition 1.1 (7.3, Proposition 22, [3]).

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G(W) \cong \bigoplus_x \mathrm{Ind}_{H_s}^K(W_s)$$

where s ranges over a set of double coset representative $S \cong K\backslash G/H$.

Proposition 1.2. *Suppose $F = \mathbb{C}$, W an FH-module and $V = \mathbb{C}G \otimes_{\mathbb{C}H} W$. Then*

$$\chi_V(x) = \frac{1}{|H|}\Sigma\chi_W(g^{-1}xg) \tag{1}$$

where the sum is over all $g \in G$ such that $g^{-1}xg \in H$.

Proof. See [3] P.30. It can also be shown using Proposition 1.1. □

Remark (Define Res_K^G and Ind_H^G for class functions). *If we replace χ_W in (1) by arbitrary class function on H, we still get a class function on G. Hence one may use (1) to **define** Ind_H^G on class functions. Similarly, Res_K^G also extends to class functions.*

Monomial Representations

Definition 1.2. *A representation is called **monomial** if it is induced from a one-dimensional representation $V = FG \otimes_{FH} W$ for some H with $\dim W = 1$.*

Lemma 1.1. *Any irreducible representation V of a p-group G over a field F of characteristic p is trivial.*

Proof. Consider G acts on $V \setminus \{0\}$ which is a set with $p^n - 1$ elements for some n. For any $y \in V \setminus \{0\}$, the cardinality of the orbit $\mathcal{O}_y$ of y must divide $|G'|$, hence is a power of p, by Orbit-Stablizer Theorem. Thus there is at least one orbit $\mathcal{O}_z$ of size 1. The one dimensional subspace V_z generated by z gives a sub-representation of V that is the trivial representation on V_z. Since V is irreducible, $V = V_z$ hence V is the trivial representation. □

Proposition 1.3. *If F is algebraically closed and G is nilpotent, every irreducible FG-module V is monomial.*

Proof. If char$F \nmid |G|$, then the proof from Theorem 16, P.66, [3] (proved for $\mathbb{C}$) carries over since nilpotent group is also supersolvable. Now suppose char$F = p$ and $p \mid |G|$. Since G is nilpotent, G is the direct product of its r-Sylow subgroups for all primes $r \mid |G|$. Write $G = H_p \times K$ where H_p is the p-Sylow subgroup and $p \nmid |K|$. Since H_p is a p-group, the restriction of V to H_p is the trivial representation by Lemma 1.1. Hence $V = \pi^* W$ where W is the FK-module given by restricting V to K and $\pi : G \to K$ is the projection. Apply the case for char$F \nmid |K|$, we finish the proof. □

Definition 1.3. *We say that G is r-**elementary**, r a prime, if G is the product of r-group and a cyclic group of order prime to r. We say G is **elementary** if G is r-elementary for some prime r.*

Remark. *Note that an elementary group is nilpotent.*

1.2 Brauer Induction and Green's Theorem

Theorem 1.1 (Brauer Induction, 10.5 Theorem 19, [3])**.** *Let G be a finite group, F an algebraically closed field, and let X denote the set of elementary subgroups of G. Then*

$$\mathrm{Ind} : \bigoplus_{H \in X} R_F H \to R_F G$$

is surjective.

Remark. *In our application to the proof of Green's theorem 1.2, we will only use the case $F = \mathbb{C}$. Note that the theorem says that every representation of G over $\mathbb{C}$ is a $\mathbb{Z}$-linear combination of representations induced from elementary subgroups.*

Corollary 1.1. *In $R_F(G)$, every representation is a $\mathbb{Z}$-linear combination of monomial representations.*

Proof. Apply Proposition 1.3 and Theorem 1.1. □

Corollary 1.2. *Let $F = \mathbb{C}$, f a class function on G. Then f is a virtual character if and only if the restriction of f to each elementary subgroup H is a virtual character.*

Proof. Let $C(G)$ be the ring of class functions. By Theorem 1.1, we can write $1 \in R_{\mathbb{C}} G \subset C(G)$ as $1 = \Sigma \, \mathrm{Ind}_{H_\alpha}^G \chi_\alpha$ where the H_α are elementary subgroups and χ_α is a virtual character of H_α. So

$$f = f \cdot 1 = \Sigma f \cdot (\mathrm{Ind}_{H_\alpha}^G)\chi_\alpha = \Sigma \, \mathrm{Ind}_{H_\alpha}^G (f|_{H_\alpha} \cdot \chi_\alpha).$$

Here, one uses Remark 1.1 for the definition of induction on class functions and a calculation using (1) gives the last equality. If $f|_{H_\alpha}$ is a virtual character, then the sum on the right hand side is a virtual character hence f is a virtual character. The other direction is clear. □

Let $\mathbb{F}_q$ denote the finite field with q elements, where $q = p^d$, p a prime. Recall that the character χ_V of a complex representation V is defined by $\chi_V(g) = \mathrm{trace}(\phi_V(g))$ where $\phi_V(g) : G \to \mathrm{Aut}\, V$. Representations are determined up to isomorphism by their characters. And $V \mapsto \chi_V$ defines an injective ring homomorphism from $R_F G$ to the ring $C(G)$ of complex-valued class functions on G. One can define characters of representations over $\mathbb{F}_q$

in the same way, but they are not so interesting. For example, the $n \times n$ identity matrix over $\mathbb{F}_q$ has trace zero if $p \mid n$. Instead we proceed as follows. Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, and fix a group embedding $i : \overline{\mathbb{F}}_q^\times \to \mathbb{C}^\times$. ($\overline{\mathbb{F}}_q^\times$ is isomorphic to the group of roots of unity in $\mathbb{C}$ which have order prime to p.)

Definition 1.4. *Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, and fix an embedding $i : \overline{\mathbb{F}}_q^\times \to \mathbb{C}^\times$. Suppose W is a representation of G over $\mathbb{F}_q$. The **Brauer character** χ_W is defined by $\chi_W(g) = \Sigma_{\alpha \in S_g} i(\alpha)$, where $S_g \subset \overline{\mathbb{F}}_q$ is the set of eigenvalues (with multiplicity) of $\phi_W(g) \in \operatorname{Aut} W$. Then χ_W is a class function on G.*

Brauer Character of Monomial Representations

Before starting the proof of next theorem, we'd like to observe some behaviors of χ_V with respect to induction. Suppose V is a monomial representation, say $V = \overline{\mathbb{F}}_q G \otimes_{\overline{\mathbb{F}}_q H} W$ with $\dim W = 1$. Then $W = \overline{\mathbb{F}}_q$ with the representation of H given by some homomorphism $\lambda : H \to \overline{\mathbb{F}}_q^\times$. Let $\tilde{\lambda} = i\lambda$ where $i : \overline{\mathbb{F}}_q^\times \hookrightarrow \mathbb{C}^\times$ is some fixed group embedding. Then $\tilde{\lambda}$ defines a representation $\tilde{W}$ of H over $\mathbb{C}$ with $\dim \tilde{W} = 1$ and $\chi_W = \chi_{\tilde{W}}$. Let $\tilde{V} = \mathbb{C}G \otimes_{\mathbb{C}H} \tilde{W}$. It's not true in general that $\chi_V = \chi_{\tilde{V}}$. We have the following example.

Example 1.1. *Let G be a p-group and $H = \{1\}$, let $\rho : G \to \mathrm{GL}(\overline{\mathbb{F}}_q)$ and $\tilde{\rho} : G \to \mathrm{GL}(\mathbb{C})$ be the representations of V and $\tilde{V}$ respectively. Then since $p \mid |G|$, the representation of V of G over $\mathbb{F}_q$ is trivial by Lemma 1.1. However, the representation $\tilde{V}$ as a representation over $\mathbb{C}$ induced from the trivial representation of H is the regular representation, i.e. the representation acting as permutation on the group $G/H \cong G$ itself. These two representations have different characters, $\chi_V \neq \chi_{\tilde{V}}$.*

Theorem 1.2 (Green)**.** *Let V be a representation of G over $\mathbb{F}_q$, then χ_V [Definition 1.4] is a virtual complex character.*

Proof. We consider several cases:

Case 1: G is a nilpotent group of order prime to p. We can assume V is irreducible, and hence monomial by Proposition 1.3, say $V = \bar{\mathbb{F}}_q G \otimes_{\bar{\mathbb{F}}_q H} W$ with $\dim W = 1$. Then $W = \bar{\mathbb{F}}_q$ with H acting via some homomorphism $\lambda : H \to \bar{\mathbb{F}}_q^\times$. Let $\tilde{\lambda} = i\lambda$ where $i : \bar{\mathbb{F}}_q^\times \subset \mathbb{C}^\times$ is a fixed embedding. Then $\tilde{\lambda}$ defines a representation $\tilde{W}$ of H over $\mathbb{C}$ with $\dim \tilde{W} = 1$ and $\chi_W = \chi_{\tilde{W}}$. Let $\tilde{V} = \mathbb{C}G \otimes_{\mathbb{C}H} \tilde{W}$.

Claim. $\chi_V = \chi_{\tilde{V}}$.

Proof of Claim. We want to show that $\chi_V(g) = \chi_{\tilde{V}}(g)$, $\forall g \in G$, so we restrict to the cyclic subgroup $K = \langle g \rangle$. By Proposition 1.1, restricted to K, both V and $\tilde{V}$ split as direct sums of K-modules U and $\tilde{U}$ respectively, where U is of the form $U = \overline{\mathbb{F}}_q K \otimes_{\overline{\mathbb{F}}_q K_s} \overline{\mathbb{F}}_q$ for $K_s = \langle g^d \rangle$ a sub-representation of K. Thus it suffices to show $\chi_U(g) = \chi_{\tilde{U}}(g)$. Suppose the representation of K_s is given by $\rho_s : K_s \to \mathrm{GL}(\overline{\mathbb{F}}_q)$ and the representation of U is given by $\rho_U : K \to \mathrm{GL}(\overline{\mathbb{F}}_q)$, let $\alpha = \rho_s(g^d)$, then the characteristic polynomial of $\rho_U(g)$ is given by $X^d - \alpha$. Since by assumption, d is prime to p, there are d distinct roots of $X^d - \alpha = 0$ in $\overline{\mathbb{F}}_q^\times$, say $\alpha_1, \ldots, \alpha_d$, thus $\chi_U(g) = \Sigma_k i(\alpha_k)$. On the other hand, the characteristic polynomial of $\rho_{\tilde{U}}(g)$ is $X^d - i(\alpha)$ which has roots $i(\alpha_1), \ldots, i(\alpha_d)$ since i is a group homomorphism with respect to multiplication. So $\chi_V(g) = \chi_{\tilde{V}}(g)$.

Case 2: G is an arbitrary nilpotent group. Again, we may assume V is irreducible. Write $G = G_p \times G'$ as a product of a p-group G_p and a group G' of order prime to p.

By the above, we have that V restricted to H is the trivial representation. Thus there is some $\overline{\mathbb{F}}_q G'$-module W such that $V = \pi^* W$ where $\pi : G \to K$ is the projection. Case 2 follows from case 1.

Case 3: G is arbitrary. By case 2, since elementary groups are nilpotent, the restriction of χ_V to each elementary subgroup H [Definition 1.3] is a virtual character of H. So by Corollary 1.2, χ_V is a virtual character of G. □

2 K-theory of Finite Fields and Brauer Lifting

In this section, we will briefly introduce higher K-theory of a ring and the construction of the Brauer lift using Green's theorem [Theorem 1.2]. The Brauer lift is an important construction that allows one to identify $BGL\mathbb{F}_q^+$ with the homotopy fibre of the map $BU \xrightarrow{\psi_q - 1} BU$ where ψ^q is the map given by Adams operation [II.4, [4]].

2.1 K-theory of Finite Fields

Let R be an associative ring with unit. Identifying each $n \times n$ matrix g with the larger matrix $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ gives an embedding of $GL_n(R)$ into $GL_{n+1}(R)$. The union of the resulting sequence $GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \dots$ is called the infinite general linear group $GL(R)$.

For any group G, we can naturally construct a connected topological space BG, called the **classifying space of** G whose fundamental group is G, but whose higher homotopy groups are zero. Details of this construction can be found in IV. 3, [4]. Moreover, the homology of the topological space BG (with coefficients in a G-module M) coincides with the algebraic homology of the group G (with coefficients in M). For $G = GL(R)$ we obtain the space $BGL(R)$. From this space, we make a further construction:

Definition 2.1 (BGL^+ for rings)**.** *[IV, Def 1.1, [4]] The notation $BGL(R)^+$ will denote any CW complex X which has a distinguished map $BGL(R) \to BGL(R)^+$ such that the following are true:*

(1) $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$, (i.e. the map $BGLR \to BGLR^+$ is the abelianization on π_1);

(2) $H^*(BGL(R); M) \xrightarrow{\cong} H^*(BGL(R)^+; M)$ for every $K_1(R)$-module M.

Definition 2.2. *For $n \geq 1$, $K_n(R)$ is defined to be the homotopy group $\pi_n BGL(R)^+$.*

Now we state the main theorem of Quillen [2].

Theorem 2.1. *Let $\mathbb{F}_q$ denote the finite field with q elements, where $q = p^d$, p a prime.*

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i - 1 \\ 0, & n \text{ even.} \end{cases} \tag{2}$$

This theorem is deduced from an even more remarkable theorem that explicitly identifies the space $BGL\mathbb{F}_q^+$. Let ψ^q denote the Adams operation in K-theory [II.4, [4]], and let $F\psi^q$ denote the homotopy fibre of $\psi^q - 1 : BU \to BU$. Here, BU is the classifying space for complex vector bundles and can also be viewed as $BGL(\mathbb{C})$.

From the long exact sequence of the fibre sequence $F\psi^q \to BU \xrightarrow{\psi^q - 1} BU$ and using that the Adams operation ψ^q on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$ is multiplication by k^i, the homotopy groups of $F\psi^q$ is can be easily calculated as following

$$\pi_n(F\psi^q) = \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i - 1 \\ 0, & n \text{ even.} \end{cases} . \tag{3}$$

Next we want to construct a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ [Section 3, [1]] and show that this map is a homotopy equivalence.

2.2 Brauer Lifting

We now construct the Brauer lift $\bar{\theta} : BGL\mathbb{F}_q^+ \to BU$.

By Green's Theoerm [Theorem 1.2], for any finite group G and any representation of G over $\mathbb{F}_q$, the Brauer character is a virtual complex character, that is, the character of a virtual complex representation. Since representation is uniquely determined by its character up to isomorphism, this gives a map between representation rings $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$. In the case of characteristic 0, any virtual complex character corresponds to a complex representation.

Take $G = GL_n(\mathbb{F}_q)$, consider the Brauer character χ_n of the standard representation $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. This gives a map $GL_n(\mathbb{F}_q) \to GL(\mathbb{C})$ hence a map $BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$. Since the characters are compatible ($\chi_n|_{GL_{n-1}} = \chi_{n-1}$), the family of maps $BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$ are compatible hence induce a map $BGL(\mathbb{F}_q) \to BGL(\mathbb{C}) = BU$. And by universal property of of $BGL(\mathbb{F}_q)^+$, this gives a map $\bar{\theta} : BGL\mathbb{F}_q^+ \to BU$.

Then we show that this map composed with $\psi^q - 1$ is nullhomotopic hence induces a map to its homotopy fibre $F\psi^q$, call this map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. Both of these spaces are H-spaces. Therefore, to show that θ is a homotopy equivalence, we show that it induces isomorphisms in homology groups. (This is referred to as Whitehead's thoerem for H-spaces. Another way to see this is using the universal property of the relative + construction [IV, Theorem 1.5, [4]] and that a homology isomorphism between H-spaces $f : X \to Y$ exhibits Y as a +-construction of X relative to the trivial subgroup of $\pi_1(X)$.)

For more details about the proof that θ induces an isomorphism in homology, see [2], [1].

References

[1] Mitchell, S. Notes on K theory of finite fields. Available online: https://sites.math.northwestern.edu/ jnkf/Mitchell-finitefieldsKtheory.pdf

[2] Quillen, Daniel. "On the cohomology and K-theory of the general linear groups over a finite field." *Annals of Mathematics 96.3* (1972): 552-586.

[3] Serre, Jean-Pierre. *Linear representations of finite groups.* Vol. 42. New York: springer, 1977.

[4] Weibel, Charles A. *The K-book: An introduction to algebraic K-theory.* Vol. 145. Providence, RI: American Mathematical Society, 2013.